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Journal of Geometry and Physics 56 (2006) 1433–1472
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

New strings for old Veneziano amplitudes III. Symplectic treatment

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Received 26 February 2005; received in revised form 28 September 2005; accepted 30 September 2005
Available online 28 November 2005

Abstract

A d-dimensional rational polytope $\mathcal{P}$ is a polytope whose vertices are located at the nodes of $\mathbf{Z}^d$ lattice. Consider the number $|k\mathcal{P} \cap \mathbf{Z}^d|$ of points inside the inflated $\mathcal{P}$ with coefficient of inflation $k(k = 1, 2, 3, \ldots)$. The Ehrhart polynomial of $\mathcal{P}$ counts the number of such lattice points inside the inflated $\mathcal{P}$ and (may be) at its faces (including vertices). In Part I [A.L. Kholodenko, New string for old Veneziano amplitudes. I. Analytical treatment, J. Geom. Phys. 55 (2005) 50–74] of our four parts work we noticed that Veneziano amplitude is just the Laplace transform of the generating function (considered as a partition function in the sense of statistical mechanics) for the Ehrhart polynomial for the regular inflated simplex obtained as deformation retract of the Fermat (hyper) surface living in the complex projective space. This observation is sufficient for development of new symplectic (this work) and supersymmetric (Part II) physical models reproducing the Veneziano (and Veneziano-like) amplitudes. General ideas (e.g. those related to the properties of Ehrhart polynomials) are illustrated by simple practical examples (e.g. use of mirror symmetry for explanation of available experimental data on $\pi\pi$ scattering, etc.) worked out in some detail. Obtained final results are in formal accord with those earlier obtained by Vergne [M. Vergne, Convex polytopes and quanization of symplectic manifolds, Proc. Natl. Acad. Sci. 93 (1996) 14238–14242].

MSC: 81T30; 13A50; 20F55; 20G45; 50B20

JGP SC: String theory; Torus actions on symplectic manifolds; Combinatorics of polytopes; Weyl character formula

Keywords: Veneziano and Veneziano-like amplitudes; Ehrhart polynomial; Toric varieties and fans; Reflexive polytopes; Mirror symmetry; Coadjoint orbits; Moment maps; Duistermaat–Heckman formula; Khovanskii–Pukhlikov correspondence

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doi:10.1016/j.geomphys.2005.09.007

1. Introduction

1.1. Connection with earlier work

In our earlier works, Refs. [1,2], which we shall call Parts I and II,[1] we initiated development of new formalism reproducing both the Veneziano and Veneziano-like (tachyon-free) amplitudes and models generating these amplitudes. In particular, in Part II we discussed one of such models. Contrary to traditional treatments, we demonstrated that our model is supersymmetric and finite-dimensional. This result was obtained with help of the theory of invariants of pseudo-reflection groups. The partition function, Eq. (II, 6.10), for this model is given by the Poincare′ polynomial

$$P((S(V)\otimes E(V))^G;z)=\prod_{i=1}^{n}\frac{1-z^{q+i}}{1-z^i}. \tag{1.1}$$

In the limit: $z\to 1$, the above result is reduced to

$$P((S(V)\otimes E(V))^G;z=1)=\frac{(q+1)(q+2)\cdots(q+n)}{n!}\equiv p(q,n). \tag{1.2}$$

which is Eq. (II, 6.11). The detailed combinatorial explanation of these results was given already in Part II. In this work, to avoid repetitions, we would like to extend such an explanation having in mind development of the symplectic model reproducing Veneziano amplitudes.

Steps toward designing of such a model were made already in Part I where it was noticed that the unsymmetrized Veneziano amplitude is obtainable as the Laplace transform of the partition function

$$P(q,t)=\sum_{n=0}^{\infty}p(q,n)t^n \tag{1.3}$$

where $p(q,n)$ is the same as in Eq. (1.2). In Ref. [3], Vergne demonstrated (without reference to string theory or Veneziano amplitudes) that such partition function has both symplectic and quantum mechanical meaning: the quantity $p(q,n)$ is dimension of the quantum Hilbert space associated (through the coadjoint orbit method) with the classical system made out of *finite* number of harmonic oscillators living on a specially designed symplectic manifold.

In this work using different arguments we reobtain her final results. Our use of different arguments is motivated by our desire to demonstrate connections between the formalism developed in this paper and that already in use in the mathematical physics literature. More importantly, the treatment presented below complements that developed earlier in Parts I and II.

In Part II, following work by Lerche et al. [4], we adopted the idea that *any* kind of one variable Poincare′ polynomial (actually, up to a constant) can be interpreted as the Weyl character formula. Since, according to Part II, both Eqs. (1.1) and (1.3) are Poincare′ polynomials, their interpretation in terms of the Weil character formula provides major ingredient toward reconstruction of the Veneziano amplitudes from the underlying quantum mechanical partition function (the Weyl character formula). Going into opposite direction, such amplitudes acquire some topological meaning to be further illuminated in this work. Direct link between topology (the Poincare′ polynomials) and quantum mechanics (the Weyl character formula) is certainly not limited to its use only for the Veneziano amplitudes and is of independent interest. In view of this, in the

[1] In referring to the results of these papers we shall use notations like Eq. (II, 5.10), etc.

next subsection we would like to provide simple arguments (different from those in the work by Lerche et al) explaining why this is so.

1.2. A motivating example

Consider a finite geometric progression of the type

$$\begin{aligned}\mathcal{F}(c,m) &= \sum_{l=-m}^{m}\exp\{cl\} = \exp\{-cm\}\sum_{l=0}^{\infty}\exp\{cl\} + \exp\{cm\}\sum_{l=-\infty}^{0}\exp\{cl\} \\ &= \exp\{-cm\}\frac{1}{1-\exp\{c\}} + \exp\{cm\}\frac{1}{1-\exp\{-c\}} \\ &= \exp\{-cm\}\left[\frac{\exp\{c(2m+1)\}-1}{\exp\{c\}-1}\right]. \end{aligned} \tag{1.4}$$

The reason for displaying the intermediate steps will become apparent shortly. First, however, we would like to consider the limit: $c \to 0^+$ of $\mathcal{F}(c,m)$. Clearly, it is given by $\mathcal{F}(0,m) = 2m+1$. The number $2m+1$ equals to the number of integer points in the segment $[-m,m]$ *including boundary* points. It is convenient to rewrite the above result in terms of $x = \exp\{c\}$. We shall write formally $\mathcal{F}(x,m)$ instead of $\mathcal{F}(c,m)$ from now on. Using these notations, let us consider the related function:

$$\bar{\mathcal{F}}(x,m) = (-1)\mathcal{F}\left(\frac{1}{x},-m\right). \tag{1.5}$$

Such type of relation (the *Ehrhart–Macdonald reciprocity law*) is characteristic for the Ehrhart polynomial for the rational polytopes to be defined in the next subsection. In Ref. [5], Stanley provides many applications of this reciprocity law. In our case, we obtain explicitly:

$$\bar{\mathcal{F}}(x,m) = (-1)\frac{x^{-(-2m+1)}-1}{x^{-1}-1}x^m. \tag{1.6}$$

In the limit $x \to 1+0^+$ we obtain: $\bar{\mathcal{F}}(1,m) = 2m-1$. The number $2m-1$ is equal to the number of integer points strictly *inside* the segment $[-m,m]$. These, seemingly trivial, results can be broadly generalized. First, we replace x by $\mathbf{x}$. Next, we replace the summation sign in the left-hand side of Eq. (1.4) by the multiple summation, etc. Thus, obtained function $\mathcal{F}(\mathbf{x},m)$ in the limit $x_i \to 1+0^+$, $i = 1-d$, produces the anticipated result:

$$\mathcal{F}(\mathbf{1},m) = (2m+1)^d \tag{1.7}$$

for number of points inside and at the edges of the d-dimensional cube in Euclidean space $\mathbf{R}^d$. Accordingly, for the number of points strictly inside the cube we obtain: $\bar{\mathcal{F}}(1,m) = (2m-1)^d$. The rationale for describing this limiting procedure is caused by its connection with our earlier result, Eq. (1.2). To explain this we need to extend our simple results in order to describe analogous situation for arbitrary centrally symmetric polytope. This is accomplished in several steps. We begin with some definitions.

Definition 1.1. A subset of $\mathbf{R}^n$ is a *polytope* (*or polyhedron*) $\mathcal{P}$ if there is a $r \times d$ matrix $\mathbf{M}$ (with $r \le d$) and a vector $\mathbf{b} \in \mathbf{R}^d$ such that

$$\mathcal{P} = \{\mathbf{x} \in \mathbf{R}^d \mid \mathbf{Mx} \le \mathbf{b}\}. \tag{1.8}$$

Definition 1.2. Provided that the Euclidean d-dimensional scalar product is given by

$$\langle \mathbf{x} \cdot \mathbf{y} \rangle = \sum_{i=1}^{d} x_i y_i \tag{1.9}$$

[2]a *rational* (*respectively*, *integral*) polytope (*or polyhedron*) $\mathcal{P}$ is defined by the set

$$\mathcal{P} = \{\mathbf{x} \in \mathbf{R}^d \mid \langle \mathbf{a}_i \cdot \mathbf{x} \rangle \leq \beta_i, \quad i = 1, \dots, r\} \tag{1.10}$$

where $\mathbf{a}_i \in \mathbf{Q}^n$ and $\beta_i \in \mathbf{Q}$ for $i = 1, \dots, r$ (respectively, $\mathbf{a}_i \in \mathbf{Z}^n$ and $\beta_i \in \mathbf{Z}$ for $i = 1, \dots, r$).

Let $Vert\,\mathcal{P}$ denote the vertex set of the rational polytope, in the case considered thus far, the d-dimensional cube. Let $\{u_1^v, \dots, u_d^v\}$ be the orthogonal basis (not necessarily of unit length) made of the highest weight vectors of the Weyl–Coxeter reflection group B_d appropriate for the cubic symmetry.[3] These vectors are oriented along the positive semi axes with respect to the center of symmetry of (hyper)cube. When parallel translated to the edges ending at particular hypercube vertex $\mathbf{v}$, they can point either in or out of this vertex. In terms of these notations, the d-dimensional version of Eq. (1.4) can be rewritten now as follows:

$$\sum_{\mathbf{x} \in \mathcal{P} \cap \mathbf{Z}^d} \exp\{\langle \mathbf{c} \cdot \mathbf{x} \rangle\} = \sum_{\mathbf{v} \in Vert\,\mathcal{P}} \exp\{\langle \mathbf{c} \cdot \mathbf{v} \rangle\} \left[\prod_{i=1}^{d} (1 - \exp\{-c_i u_i^v\})\right]^{-1}. \tag{1.11}$$

Correctness of this equation can be readily checked by considering special cases of a segment, square, cube, etc. The result, Eq. (1.11), obtained for the polytope of cubic symmetry can be extended to the arbitrary convex centrally symmetric polytope as we shall demonstrate below. This fact allows us to investigate properties of more complex polytopes with help of polytopes of cubic symmetry. Moreover, we shall argue below that the right-hand side of Eq. (1.11) is mathematically equivalent to the right-hand side of Eq. (1.1). Because of this, the limiting procedure $c \to 0^+$ producing the number of points inside (and at the boundaries) of the polyhedron $\mathcal{P}$ in the left-hand side of Eq. (1.11) is of the same nature as the limiting procedure: $z \to 1$ in Eq. (1.2) where, as result of this procedure, the right-hand side of Eq. (1.2) produces the number of lattice points for the inflated (with inflation coefficient q) rational simplex of dimension n "living" in $\mathbf{Z}^n$ lattice.

Remark 1.3. For an arbitrary convex polytope the above formula, Eq. (1.11), was obtained (seemingly independently) in many different contexts. For instance, in the context of discrete and computational geometry it is attributed to Brion [6]. In view of Eq. (1.5), it can as well be attributed to Ehrhart, Stanley [5] and to many others. In fact, for the case of centrally symmetric polytopes this formula is just a special case of the Weyl's character formula. This will be demonstrated below, in Section 2.

There are many paths to arrive at final results of this paper, i.e. to reobtain the results of Vergne [3], and to use them for construction of new models reproducing the Veneziano and Veneziano-like amplitudes. They include, for instance, the algebro-geometric, symplectic, group-theoretic, combinatorial, supersymmetric, etc. pathways to reach the same destination. In our opinion, the most direct way to arrive at final results is combinatorial. Although it will be discussed at length in Part IV from yet another perspective, in this work we present some essentials needed for their

[2] So that $\mathbf{x}$ lives in space *dual* to that for $\mathbf{y}$.

[3] For a brief guide to the Weyl–Coxeter reflection groups, please, see Appendix A to Part II.

immediate use in the rest of the paper. In particular, we would like to discuss now some facts about the Ehrhart polynomials having in mind their uses in high energy physics.

1.3. Ehrhart polynomials, mirror symmetry and the extended Veneziano amplitudes

In the previous subsection we introduced Eq. (1.5). It characterizes the Ehrhart polynomial. It is important to realize that $p(q, n)$ in Eq. (1.2) already is an example of the Ehrhart polynomial. Evidently, Eq. (1.2) can be written formally as

$$p(q, n) = a_n q^n + a_{n-1} q^{n-1} + \cdots + a_0. \tag{1.12}$$

In Ref. [7], it is argued that for *any* convex rational polytope $\mathcal{P}$ the Ehrhart polynomial can be written as

$$|q\mathcal{P} \cap \mathbf{Z}^n| = \mathfrak{P}(q, n) = a_n(\mathcal{P}) q^n + a_{n-1}(\mathcal{P}) q^{n-1} + \cdots + a_0(\mathcal{P}) \tag{1.13}$$

with coefficients $a_0, \ldots, a_n$ specific for a given polytope $\mathcal{P}$. Nevertheless, irrespective to the type of polytope $\mathcal{P}$, it is known that $a_0 = 1$ and $a_n = Vol\mathcal{P}$, where $Vol\mathcal{P}$ is the *Euclidean* volume of the polytope. To calculate the remaining coefficients of this polynomial explicitly for an arbitrary convex polytope is a difficult task in general. Such task was accomplished rather recently in Ref. [8]. The authors of [8] recognized that in order to obtain the remaining coefficients it is useful to calculate the generating function for the Ehrhart polynomial. In our case this function is given by Eq. (1.3). From Eq. (I, 1.22) we already know that formally this is the partition function for the unsymmetrized Veneziano amplitude. In view of our Eq. (1.1) taken from Part II, we also know that it can be also looked upon as the partition function for the Veneziano amplitudes. Hence, now we would like to explain how Eqs. (1.1) and (1.3) are related to each other from the point of view of commutative algebra and combinatorics of polytopes. By doing so some useful physical information will be obtained as well.

Long before the results of Ref. [8] were published, it was known [9] that the generating function for the Ehrhart polynomial of $\mathcal{P}$ can be written in the following universal form

$$\mathcal{F}(\mathcal{P}, x) = \sum_{q=0}^{\infty} \mathfrak{P}(q, n) x^q = \frac{h_0(\mathcal{P}) + h_1(\mathcal{P})x + \cdots + h_n(\mathcal{P})x^n}{(1-x)^{n+1}}. \tag{1.14}$$

For the Veneziano partition function all coefficients, except $h_0(\mathcal{P})$, are zero and, of course, $h_0(\mathcal{P}) = 1$ [10]. This can be easily understood in view of Eq. (I, 1.20). We brought to our readers attention the above general result in view of our task of comparing Eqs. (1.1) and (1.3) and of possibly generalizing the Veneziano amplitudes and the partition functions associated with them.

In practical applications it should be noted that the combinatorial factor $p(q, n)$, Eq. (1.2), representing the number of points in the inflated simplex $\mathcal{P}$ (with coefficient of inflation q) whose vertex set *Vert* $\mathcal{P}$ belongs to $\mathbf{Z}^n$ lattice can be formally written in several equivalent ways. In particular, as we have mentioned in Part II,

$$p(q, n) = \frac{(q+n)!}{q!n!} = \frac{(q+1)\cdots(q+n)}{n!} = \frac{(n+1)\cdots(n+q)}{q!}. \tag{1.15}$$

This fact has some physical significance. For instance, in particle physics literature, e.g. see Refs. [11,12], the third option is commonly used. Let us recall how this happens. One is looking for an expansion of the factor $(1-x)^{-\alpha(t)-1}$ under the integral of beta function as explained in Part I. Looking at Eq. (1.14) one realizes that the Mandelstam variable $\alpha(t)$ plays a role of dimensionality of **Z**-lattice. Hence, we have to identify it with n in the *second option* provided by Eq. (1.15). This is not the way such an identification is done in physics literature where, in fact, the third option provided by Eq. (1.15) is commonly used with $n = \alpha(t)$ effectively being the inflation factor while q effectively being the dimensionality of the lattice.[4] A quick look at Eqs. (1.3) and (1.14) shows that under such circumstances the generating function for the Ehrhart polynomial and that for the Veneziano amplitude are formally *not* the same: in the first (mathematical) case one is dealing with lattices of fixed dimensionality and is considering summation over various inflation factors at the same time, while in the third (physical) case, one is dealing with the fixed inflation factor $n = \alpha(t)$ while summing over lattices of different dimensionalities. Such arguments are superficial however in view of Eqs. (I, 1.20) and (1.14) above. Using these equations it is clear that correct agreement between Eqs. (1.3) and (1.14) can be reached if one is using $\mathfrak{P}(q, n) = p(q, n)$ with the second (i.e. mathematical) option offered by Eq. (1.15). By doing so no changes in the pole locations for the Veneziano amplitude occur. Moreover, for a given pole the second and the third option in Eq. (1.15) produce exactly the same contributions into the residue thus making them *physically* indistinguishable. Nevertheless, our choice of the mathematically meaningful interpretation of the Veneziano amplitude as the Laplace transform of the Ehrhart polynomial generating function provides one of the *major reasons* for development of the formalism of Parts I through IV. In particular, it allows us to think about possible generalizations of the Veneziano amplitude using generating functions for the Ehrhart polynomials for polytopes of other types. As it is demonstrated by Stanley [9,13], both Eqs. (1.1) and (1.14) have group—invariant meaning as Poincare$'$ polynomials: Eq. (1.14) is associated with the Poincare$'$ polynomial for the so called Stanley–Reisner polynomial ring while Eq. (1.1) is the Poincare polynomial for the so-called Gorenstein ring. Naturally, these two rings are interrelated thus providing the desired connection between Eqs. (1.1) and (1.14). For the sake of space, we refer our readers to the original works by Stanley [9,13] where all mathematical details can be found. At the same time, we have supplied sufficient information in order to discuss some physical applications. In particular, following Batyrev [14, p. 392], and Hibi [15], we would like to discuss the reflexive (polar (or dual)) polytopes playing major role in calculations involving mirror symmetry. To those of our readers who are familiar with some basic facts of solid state physics [16] the concept of a dual (or polar) polytope should look quite familiar since it is completely analogous to that for the reciprocal lattice. Both direct and reciprocal lattices are used routinely in calculations related to physical properties of crystalline solids. The requirement that physical observables should remain the same irrespective to what lattice is used in computations is completely natural. Not surprisingly, such a requirement formally coincides with that used in the high energy physics. In the paper by Greene and Plesser [17, p. 26], one finds the following statement: "Thus, we have demonstrated that two topologically distinct Calabi–Yau manifolds M and M' give rise to the same conformal field theory. Furthermore, although our argument has been based only at one point in the respective moduli spaces $\mathcal{M}_M$ and $\mathcal{M}_{M'}$ of M and M'(namely the point which has a minimal model interpretation and hence respects the symmetries by which we have orbifolded) the result necessarily extends to all of $\mathcal{M}_M$ and $\mathcal{M}_{M'}$". In Parts I and II we argued that, in view of the Veneziano condition, there is a significant difference between calculations of observables (amplitudes) of high energy physics and those in conformal field theories. This difference is analogous to the difference between the point group symmetries in the liquid/gas phases and translational symmetries of solid phases. Hence, extension of the Veneziano amplitudes with

[4] We have to warn our readers that, to our knowledge, nowhere in physics literature such combinatorial terminology is being used.

help of general result, Eq. (1.14) (which is essentially equivalent to accounting for the mirror symmetry) requires some explanations. We would like to provide a sketch of these explanations now.[5]

To this purpose we need to introduce several definitions first. We begin with

Definition 1.4. For any convex polytope $\mathcal{P}$ the dual polytope $\mathcal{P}^*$ is defined by

$$\mathcal{P}^* = \{\mathbf{x} \in (\mathbf{R}^d)^* \mid \langle \mathbf{a} \cdot \mathbf{x} \rangle \leq 1, \quad \mathbf{a} \in \mathcal{P}\} \tag{1.16}$$

Although in algebraic geometry of toric varieties the inequality $\langle \mathbf{a} \cdot \mathbf{x} \rangle \leq 1$ sometimes is replaced by $\langle \mathbf{a} \cdot \mathbf{x} \rangle \geq -1$ [10], we shall use Eq. (1.16) to be in accord with Hibi [15]. According to this reference, if $\mathcal{P}$ is rational, then $\mathcal{P}^*$ is also rational. However, $\mathcal{P}^*$ is not necessarily integral even if $\mathcal{P}$ is integral. This fact is profoundly important since the result, Eq. (1.14), is valid for the integral polytopes only. The question arises: under what conditions is the dual polytope $\mathcal{P}^*$ integral? The answer is given by the following.

Theorem 1.5 (Hibi [15]). *The dual polytope $\mathcal{P}^*$ is **integral** if and only if*

$$\mathcal{F}(\mathcal{P}, x^{-1}) = (-1)^{d+1} x \mathcal{F}(\mathcal{P}, x) \tag{1.17}$$

where the generating function $\mathcal{F}(\mathcal{P}, x)$ is defined earlier by Eq. (1.14).

By combining Eqs. (1.3) and (1.14) we obtain for the standard Veneziano amplitude the following result:

$$\mathcal{F}(\mathcal{P}, x) = \left(\frac{1}{1-x}\right)^{d+1}. \tag{1.18}$$

Using it in Eq. (1.17) produces:

$$\mathcal{F}(\mathcal{P}, x^{-1}) = \frac{(-1)^{d+1}}{(1-x)^{d+1}} x^{d+1} = (-1)^{d+1} x^{d+1} \mathcal{F}(\mathcal{P}, x). \tag{1.19}$$

This result indicates that scattering processes described by the standard Veneziano amplitudes *do not* involve any mirror symmetry since, as it is well known, Refs. [14,18], in order for such a symmetry to take place the dual polytope $\mathcal{P}^*$ *must* be integral. The question arises: can these amplitudes be modified with help of Eq. (1.14) so that mirror symmetry can be in principle observed in nature? To answer this question, let us assume that, indeed, Eq. (1.14) can be used for such a modification. In this case we obtain

$$\mathcal{F}(\mathcal{P}, x^{-1}) = (-1)^{d+1} x \mathcal{F}(\mathcal{P}, x) \tag{1.20}$$

provided that $h_{n-i} = h_i$ in Eq. (1.14). But this is surely the case in view of the fact that these are the Dehn–Sommerville equations, Ref. [10, p. 16]. Hence, at this stage of our discussion, it looks like generalization of the Veneziano amplitudes which takes into account mirror symmetry is possible from the mathematical standpoint. Mathematical arguments themself are *not sufficient* however for such generalization. This is so because of the following chain of arguments.

Already in the original paper by Veneziano [19, p. 195], it was noticed that the amplitude he defined originally is not unique. Following Ref. [20, p. 100], we notice that beta function $B(-\alpha(s), -\alpha(t))$ in Veneziano amplitude can be replaced by

[5] To keep focus of our readers on major issues of this paper, we suppress to the absolute minimum the discussion connecting our results to experiment. We plan to discuss this connection thoroughly in a separate publication.

$$B(m-\alpha(s), n-\alpha(t)) \tag{1.21}$$

for any integers $m, n \geq 1$, and similarly for s, u and t, u terms. According to Ref. [20], "Any function which can be presented as linear combination of terms like (1.21) satisfies the finite energy sum rules, so there is no constraint on the resonance parameters *unless additional assumptions are made*." The mirror symmetry arguments presented above *are such additional assumptions*. To use them wisely, we still need to make several remarks. Experimentally, the linear combination of terms in the form given by Eq. (1.21) should show as explained in up in the form of daughter (or satellite) Regge trajectories, Refs. [20,21]. But, according to Frampton [22], such daughter trajectories should be present even for the standard (that is non generalized!) Veneziano amplitudes. This is so because of the following arguments. In accord with analysis made in Part I, the unsymmetrized Veneziano amplitude can be presented as

$$V(s,t) = \sum_{n=0}^{\infty} p(\alpha(t), n)\frac{1}{n-\alpha(s)}. \tag{1.22}$$

For a given n the polynomial $p(\alpha(t), n)$ is an n-th degree polynomial in $\alpha(t)$ (for high enough energies in t). Since in the Regge theory n represents the total spin, according to the rules of quantum mechanics, in addition to particles with spin n there should (could) be particles with spins $n-1, n-2, \ldots, 1, 0$. These particles should be visible (in principle) at the parallel (daughter or satellite) trajectories all lying below the leading (with spin n) Regge trajectory. While the leading trajectory has $\alpha(t)^n$ as its residue, the daughter trajectories have $(\alpha(t))^{n-1}$, $(\alpha(t))^{n-2}$, etc., as their residues. Unfortunately, in addition, the countable infinity of satellite (daughter) trajectories can originate if the masses of colliding particles are not the same, Ref. [20, p. 40]. The linear combination of terms given by Eq. (1.21) can account in principle for such phenomena. Following Frampton [22], we need to take into account that the linear combination of terms in Eq. (1.21) can be replaced (quite rigorously) by the combination of terms of the form

$$B(n-\alpha(s), n-\alpha(t)) \tag{1.23}$$

with $n \geq 0$. In real life the infinite number of trajectories is never observed however. But several parent–daughter Regge trajectories are being observed quite frequently, e.g., see Ref. [20, p. 41]. If we accept the existence of mirror symmetry, these observational facts can be explained quite naturally. To illustrate this, we would like to consider the simplest case of $\pi\pi$ scattering described in Refs. [12,22]. Below the threshold, that is below the collision energies producing more outgoing particles than incoming, the unsymmetrized amplitude $A(s,t)$ for such a process is known to be

$$A(s,t) = -g^2\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} = -g^2(1-\alpha(s)-\alpha(t))B(1-\alpha(s), 1-\alpha(t)). \tag{1.24}$$

This result should be understood as follows. Consider the "weighted" (unsymmetrized) Veneziano amplitude of the type

$$A(s,t) = \int_0^1 \mathrm{d}x x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1} g(x,s,t) \tag{1.25}$$

where the weight function $g(x,s,t)$ is given by

$$g(x,s,t) = \frac{1}{2}g^2[(1-x)\alpha(s) + x\alpha(t)]. \tag{1.26a}$$

Alternatively, the same result, Eq. (1.24), is obtained if one uses instead the weight function

$$g(x, s, t) = g^2 x\alpha(t). \tag{1.26b}$$

Consider now a special case of Eq. (1.14): $n = 2$. For such case we obtain

$$\mathcal{F}(\mathcal{P}, x) = \sum_{q=0}^{\infty} \mathfrak{P}(q, 1)x^q = \frac{h_0(\mathcal{P}) + h_1(\mathcal{P})x}{(1-x)^{1+1}} \tag{1.27}$$

so that Eq. (1.20) holds thus indicating presence of mirror symmetry. At this point our readers might notice that, actually, for this symmetry to take place one should consider instead of amplitude $A(s, t)$ the following combination

$$\begin{aligned} A(s, t) &= -g^2 \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} + g^2 \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \\ &= -\hat{g}^2 B(1-\alpha(s), 1-\alpha(t)) + g^2 B(-\alpha(s), -\alpha(t)). \end{aligned} \tag{1.28}$$

Such a combination produces first two terms (with correct signs) of the infinite series as proposed by Mandelstam, Eq. (15), Ref. [23]. The comparison with experiment displayed in Fig. 6.2(a), Ref. [22, p. 321] is quite satisfactory producing one parent and one daughter trajectory. These are also displayed in Ref. [20, p. 41] for the "rho family". It should be noted that in the present case the phase factors (eliminating tachyons) discussed in Part I are not displayed since in Ref. [24] and, therefore also in this work, we provide alternative explanation why tachyons should be excluded from consideration.

1.4. Organization of the rest of the paper

In Section 2, using the concept of a zonotope we prove that, indeed, Eq. (1.11) (and, hence, Eqs. (1.1) and (1.3)) is a special case of the Weyl character formula. In arriving at this result we employ some information about the Ruelle's dynamical transfer operator and earlier results by Atiyah and Bott on Lefshetz-type fixed point formula for the elliptic complexes. Section 3 along with results of Appendix A (Part II) provides necessary mathematical background to be used in Section 4. It includes some relevant facts from the theory of toric varieties, algebraic groups, semisimple Lie groups and associated with them Lie algebras, flag decompositions, etc. With help of this information in Section 4 major physical applications are developed culminating in the exact symplectic solution of the Veneziano model. Connection between the symplectic and supersymmetric formalisms was noticed and developed in the classical paper by Atiyah and Bott [25], whose work had been inspired by earlier result by Witten [26], on supersymmetry and Morse theory. Thus, in view of Ref. [25], the results of Parts II and III become interrelated. The final results obtained in this work are in formal accord with those obtained earlier by Vergne [3], by other methods.

2. From geometric progression to Weyl character formula

2.1. From p-cubes to d-polytopes via zonotope construction

In Section 1, we have obtained Eq. (1.11). In one-dimensional case it is formally reduced to a simple geometric progression formula. The result, Eq. (1.11), is obtained for the rational (or integral) polytope of cubic symmetry. In this subsection we generalize this result to obtain similar results for rational centrally symmetric polytopes whose symmetry is other than cubic. This is possible with help of the concept of zonotope. The concept of zonotope is not new. According to Coxeter [27], it belongs to the 19th century Russian crystallographer Fedorov. Nevertheless, this concept has been truly appreciated only relatively recently in connection with oriented matroids. For the purposes of this work it is sufficient to consider only the most elementary properties of zonotopes. Thus, following Ref. [28], let us consider a p-dimensional cube C_p defined by

$$C_p = \{\mathbf{x} \in \mathbf{R}^p, \quad -1 \le x_i \le 1,\ i = 1 - p\} \tag{2.1}$$

and the surjective map $\pi : \mathbf{R}^p \to \mathbf{R}^d$. The map is defined via the following.

Definition 2.1. A *zonotope* $Z(V)$ is the image of a p-cube, Eq. (2.1), under the affine projection π. Specifically:

$$Z \equiv Z(V) = \mathbf{V}C_p + \mathbf{z} = \{V\mathbf{y} + \mathbf{z} : \mathbf{y} \in C_p\} = \{x \in \mathbf{R}^d : x$$
$$= z + \sum_{i=1}^{p} x_i \mathbf{v}_i, \quad -1 \le x \le 1\}$$

for some matrix (vector configuration) $\mathbf{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

By construction, such an image is a centrosymmetric d-polytope [28]. Below, we shall obtain some results for these d-polytopes. By construction, they should hold also for p-cubes. In such a way we shall demonstrate that, indeed, Eq. (1.11) can be associated with the Weyl character formula.

2.2. From Ruelle dynamical transfer operator to Atiyah and Bott Lefschetz-type fixed point formula for the elliptic complexes

Any classical dynamical system can be thought of as the pair $(\mathcal{M}, \mathrm{f})$ with f being a map f: $\mathcal{M} \to \mathcal{M}$ from the phase space $\mathcal{M}$ to itself. Following Ruelle [29], we consider a map f: $\mathcal{M} \to \mathcal{M}$ and a scalar function (a weight function) g: $\mathcal{M} \to \mathbf{C}$. Based on these data, the transfer operator $\mathcal{L}$ can be defined as follows:

$$\mathcal{L}\Phi(x) = \sum_{y:\, fy = x} g(y)\Phi(y). \tag{2.2}$$

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two such transfer operators associated with some successive maps f_1, f_2: $\mathcal{M} \to \mathcal{M}$ and weights g_1 and g_2 then

$$(\mathcal{L}_1\mathcal{L}_2\Phi)(x) = \sum_{y:\, f_2 f_1 y = x} g_2(f_1 y) g_1(y)\Phi(y). \tag{2.3}$$

It is possible to demonstrate that

$$tr\, \mathcal{L} = \sum_{x \in Fix\, f} \frac{g(x)}{|\det(1 - D_x f^{-1}(x))|} \tag{2.4}$$

with $D_x f$ being derivative of f acting in the tangent space $T_x\mathcal{M}$ and the graph of f is required to be transversal to the diagonal $\Delta \subset \mathcal{M} \times \mathcal{M}$. Eq. (2.4) coincides with that obtained in the work by Atiyah and Bott [30].[6] In connection with Eq. (2.4), these authors make several important (for purposes of this work) observations to be discussed now. In Ref. [29] Ruelle essentially uses the same type of arguments as those by Atiyah and Bott [30]. These are given as follows. Define the *local* Lefschetz index $\mathcal{L}_x(f)$ by

$$\mathcal{L}_x(f) = \frac{\det(1 - D_x f(x))}{|\det(1 - D_x f(x))|}, \tag{2.5}$$

where $x \in Fix\ f$. Then define the *global* Lefschetz index $\mathcal{L}(f)$ by

$$\mathcal{L}(f) = \sum_{f(x)=x} \mathcal{L}_x(f). \tag{2.6}$$

Taking into account that $\det(1 - D_x f(x)) = \prod_{i=1}^{d}(1-\alpha_i)$, where α_i are the eigenvalues of the Jacobian matrix, the determinant can be rewritten in the following useful form, Ref. [31, p. 133]:

$$\det(1 - D_x f(x)) = \prod_{i=1}^{d}(1-\alpha_i) = \sum_{k=0}^{d}(-1)^k e_k(\alpha_1, \ldots, \alpha_d), \tag{2.7}$$

where the elementary symmetric polynomial $e_k(\alpha_1, \ldots, \alpha_d)$ is defined by

$$e_k(\alpha_1, \ldots, \alpha_d) = \sum_{1 \le i_1 < \cdots < i_k \le d} \alpha_{i_1}, \ldots, \alpha_{i_k} \tag{2.8}$$

with $e_{k=0} = 1$. With help of these results the local Lefschetz index, Eq. (2.5), can be rewritten alternatively as follows:

$$\mathcal{L}_x(f) = \frac{\sum_{k=0}^{d}(-1)^k e_k(\alpha_1, \ldots, \alpha_d)}{|\det(1 - D_x f(x))|} \equiv \frac{\sum_{k=0}^{d}(-1)^k tr(\wedge^k D_x f(x))}{|\det(1 - D_x f(x))|} \tag{2.9}$$

with $\wedge^k$ denoting the k-th power of the exterior product. Using this result, Ruelle [29] defined additional transfer operator $\mathcal{L}^{(k)}$ (analogous to earlier introduced $\mathcal{L}$) as follows:

$$tr\ \mathcal{L}^{(k)} = \sum_{x \in Fix\ f} \frac{g(x) tr(\wedge^k D_x f(x))}{|\det(1 - D_x f^{-1}(x))|}. \tag{2.10}$$

In view of Eqs. (2.5)–(2.10), we also obtain

$$\sum_{k=0}^{d}(-1)^k tr\ \mathcal{L}^{(k)} = \sum_{x \in Fix\ f} g(x)\mathcal{L}_x(f). \tag{2.11}$$

If in the above formulas we replace *Fix f* by $Fix\ f^n$, we have to replace $tr\ \mathcal{L}^{(k)}$ by $tr\ \mathcal{L}_n^{(k)}$. Next, since

$$\exp\left(\sum_{n=1}^{\infty} \frac{tr(\mathbf{A}^n)}{n} t^n\right) = [\det(\mathbf{1} - t\mathbf{A})]^{-1} \tag{2.12}$$

[6] They use $D_x f$ instead of $D_x f^{-1}$ which makes no difference for the fixed points and invertible functions. The important (for chaotic dynamics) non invertible case is discussed by Ruelle also but the results are not much different.

it is convenient to combine this result with Eq. (2.10) in order to obtain the following dynamical zeta function:

$$Z(t) = \exp\left(\sum_{n=1}^{\infty}\frac{t^n}{n}\left\{\sum_{k=0}^{d}(-1)^k tr\,\mathcal{L}_n^{(k)}\right\}\right) = \prod_{k=0}^{d}\left[\exp\left(\sum_{n=1}^{\infty}\frac{tr\,\mathcal{L}_n^{(k)}}{n}t^n\right)\right]^{(-1)^k}$$
$$= \prod_{k=0}^{d}[\det(\mathbf{1} - t\mathcal{L}^{(k)})]^{(-1)^{k+1}}. \tag{2.13}$$

This final result coincides with that obtained by Ruelle as required. Thus obtained zeta function possess dynamical, number-theoretic and algebro-geometric interpretation. Looking at Eq. (1.1), it should be clear that for the appropriately chosen d and $\mathcal{L}$ Eqs. (1.1) and (2.13) can be made the same. Moreover, based on the paper by Atiyah and Bott [30], we would like to demonstrate that Eq. (2.4) is actually the same thing as the Weyl's character formula [32]. To prove that this is indeed the case is not entirely trivial. In what follows, we shall assume that our readers are familiar with results and notations of Part II and, especially, with the results of Appendix A to Part II. To avoid duplications, we shall use below results on the Weyl–Coxeter reflection groups using notations from this appendix.[7]

2.3. *From Atiyah–Bott–Lefschetz fixed point formula to character formula by Weyl*

We begin with observation that Eqs. (2.4) and (1.11) are equivalent. Because of this, it is sufficient to demonstrate that the right-hand side of Eq. (1.11) indeed coincides with the Weyl's character formula. Although Eq. (1.11) (and, especially, Eqs. (2.3) and (2.10)) looks similar to that obtained in the paper by Atiyah and Bott (AB), Ref. [30, Part I, p. 379], leading to the Weyl character formula, Eq. (5.12) of Ref. [30, Part II],[8] neither Eq. (1.11) nor Eq. (5.11) of AB paper [30, Part II] provide immediate connection with their Eq. (5.12). Hence, the task now is to restore some missing links.

To this purpose we need to recall some facts from the book by Bourbaki [32]. These facts are also helpful in the remainder of this paper. In particular, let us consider a finite set of formal symbols $e(\mu)$ possessing the same multiplication properties as the usual exponents,[9] i.e.

$$e(\mu)e(\nu) = e(\mu+\nu), \quad [e(\mu)]^{-1} = e(-\mu) \text{ and } e(0) = 1. \tag{2.14}$$

Such defined set of formal exponents is making a free $\mathbf{Z}$ module with the basis $e(\mu)$. Subsequently, we shall require that $\mu \in \Delta$ with Δ being the Weyl root system defined in the Appendix. Suppose also that we are given a polynomial ring $A[\mathbf{X}]$ made of all linear combinations of terms $\mathbf{X}^{\mathbf{n}} \equiv X_1^{n_1}, \ldots, X_d^{n_d}$ with $n_i \in \mathbf{Z}$ and X_i being some indeterminates. Then, one can construct another ring A[$\mathbf{P}$] made of linear combinations of elements $e(\mathbf{p}\cdot\mathbf{n})$ with $\mathbf{p}\cdot\mathbf{n} = p_1n_1 + \cdots + p_dn_d$. Clearly, the above rings are isomorphic as it was explained in Part II, Section 9. Let $x = \sum_{p\in P} x_p e(p) \in$

[7] To shorten notations we shall write "Appendix" having in mind the Appendix A of Part II.

[8] It should be taken into account that the AB paper also has Parts I and II.

[9] In the case of *usual* exponents it is being assumed that all the properties of formal exponents are transferable to the usual ones.

$A[\mathbf{P}]$ with $\mathbf{P} = \{p_1, \ldots, p_d\}$. Then using Eq. (2.14) we obtain:

$$x \cdot y = \sum_{s\in P} x_s e(s) \sum_{r\in P} y_r e(r) = \sum_{t\in P} z_t e(t) \text{ with } z_t = \sum_{s+r=t} x_s y_r \quad \text{and, accordingly,}$$

$$x^m = \sum_{t\in P} z_t e(t) \text{ with } z_t = \sum_{s+\cdots+r=t} x_s, \ldots, y_r, m \in \mathbf{N} \tag{2.15}$$

with $\mathbf{N}$ being some non-negative integer. Introduce now the determinant of $w \in W$ via rule:

$$\det(w) \equiv \varepsilon(w) = (-1)^{l(w)}, \tag{2.16}$$

where, again, we use notations from Appendix. If, in addition, we would require

$$w(e(p)) = e(w(p)), \tag{2.17}$$

then all elements of the ring A[**P**] are subdivided into two classes defined by

$$w(x) = x(\text{invariance}) \tag{2.18a}$$

and

$$w(x) = \varepsilon(x) \cdot x(\text{anti invariance}). \tag{2.18b}$$

These classes are very much like subdivision into bosons and fermions in quantum mechanics.[10] All anti invariant elements can be built from the basic anti invariant element $J(x)$ which, in view of Eq. (2.7), can defined by

$$J(x) = \sum_{w\in W} \varepsilon(w) \cdot w(x). \tag{2.19}$$

From the definition of the set **P** and from Appendix it should be clear that the set **P** can be identified with the set of reflection elements w of the Weyl group W. Therefore, for all $x \in A[\mathbf{P}]$ and $w \in W$ we obtain the following chain of equalities:

$$w(J(x)) = \sum_{v\in W} \varepsilon(v) \cdot w(v(x)) = \varepsilon(w) \sum_{v\in W} \varepsilon(v) \cdot v(x) = \varepsilon(w) J(x) \tag{2.20}$$

as required. Accordingly, *any* anti invariant element x can be written as $x = \sum_{l\in P} x_p J(\exp(p))$. The denominator of Eq. (1.11), when properly interpreted with help of Appendix, can be associated with $J(x)$. Indeed, without loss of generality let us choose the constant $\mathbf{c}$ as $\mathbf{c} = \{1, \ldots, 1\}$. Then, for the fixed v the denominator of Eq. (1.11) can be rewritten as follows:

$$\prod_{i=1}^{d} (1 - \exp\{-u_i^v\}) \equiv \prod_{\alpha\in\Delta^+} (1 - \exp(-\alpha)) \equiv \tilde{d} \exp(-\rho), \tag{2.21}$$

where

$$\rho = \frac{1}{2} \sum_{\alpha\in\Delta^+} \alpha \quad \text{and}$$
$$\tilde{d} = \prod_{\alpha\in\Delta^+} \left(\exp\left(\frac{\alpha}{2}\right) - \exp\left(-\frac{\alpha}{2}\right) \right). \tag{2.22}$$

[10] Such analogy is not superficial as we have noticed already in Part II.

To prove that thus defined $\tilde{d}$ belongs to the anti invariant subset of $A[P]$ is not difficult. Indeed, consider the action of a reflection r_i on $\tilde{d}$. Taking into account that $r_i(\alpha_i) = -\alpha_i$ we obtain

$$r_i(\tilde{d}) = \left(\exp\left(-\frac{\alpha_i}{2}\right) - \exp\left(\frac{\alpha_i}{2}\right)\right) \prod_{\substack{\alpha\neq\alpha_i \\ \alpha\in\Delta^+}} \left(\exp\left(\frac{\alpha}{2}\right) - \exp\left(-\frac{\alpha}{2}\right)\right) = -\tilde{d} \equiv \varepsilon(r_i)\tilde{d}. \tag{2.23}$$

Hence, clearly

$$\tilde{d} = \sum_{p\in P} x_p J(\exp(p)). \tag{2.24}$$

Moreover, it can be shown [32], that $\tilde{d} = J(\exp(\rho))$ which, in view of Eqs. (2.21) and (2.22), produces identity originally obtained by Weyl:

$$\hat{d}\exp(-\rho) = \prod_{\alpha\in\Delta^+} (1 - \exp(-\alpha)). \tag{2.25}$$

Applying reflection w to the above identity while taking into account Eqs. (2.17) and (2.23) produces:

$$\prod_{\alpha\in\Delta^+} (1 - \exp(-w(\alpha))) = \exp(-w(\rho))\varepsilon(w)\hat{d}. \tag{2.26}$$

The result just obtained is of central importance for the proof of the Weyl's formula. Indeed, in view of Eqs. (2.17) and (2.26), inserting the identity $1 = \frac{w}{w}$ into the sum over the vertices on the right-hand side of Eq. (1.11) and taking into account that, (a) $\varepsilon(w) = \pm 1$ so that $[\varepsilon(w)]^{-1} = \varepsilon(w)$, (b) actually, the sum over the vertices is the same thing as the sum over the members of the Weyl–Coxeter group (since all vertices of the integral polytope can be obtained by using the appropriate reflections applied to the highest weight vector pointing to a chosen vertex), we obtain the Weyl's character formula:

$$tr\,\mathcal{L}(\lambda) = \frac{1}{\hat{d}} \sum_{w\in\Delta} \varepsilon(w)\exp\{w(\lambda+\rho)\}. \tag{2.27}$$

It was obtained with help of the results of Appendix, Eqs. (2.4), (2.17) and (2.26). Looking at the left-hand side of Eq. (1.11) we can replace $tr\,\mathcal{L}(\lambda)$ by the sum in the left-hand side of Eq. (1.11) if we choose the constant $\mathbf{c}$ as before. Doing this is not too illuminating however as we would like to explain now.

Indeed, since by construction $J(x)$ in Eq. (2.19) is the basic anti invariant element and the right-hand side of Eq. (2.27) is by design manifestly invariant element of the $A[\mathbf{P}]$, it is only natural to look for the basic invariant element analogue of $J(x)$. Then, in view of Eq. (2.15) (and discussion preceding this equation), $tr\,\mathcal{L}(\lambda) \equiv \chi(\lambda)$ should be expressible as follows:

$$\chi(\lambda) = \sum_{w\in W} n_w(\lambda)\, e(w). \tag{2.28}$$

The factor $n_w(\lambda)$ in Eq. (2.28) is known in group theory as the Kostant's multiplicity formula [33]. It plays an important role in our work, especially in Section 4. To calculate it explicitly, Cartier [34] simplified the original derivation by Kostant. In view of simplicity of his arguments, we would like to reproduce them having in mind their later use in the text. Cartier noticed that the denominator of the Weyl character formula, Eq. (2.27), can be formally expanded with help of Eq. (2.25) as follows:

$$\left[\exp(\rho)\prod_{\alpha\in\Delta^+}(1-\exp(-\alpha))\right]^{-1}=\sum_{w'\in W}P(w')e(-\rho-w'). \tag{2.29}$$

By combining Eqs. (2.27)–(2.29) we obtain,

$$\sum_{w\in W}n_w(\lambda)\,e(w)=\sum_{w\in W}\varepsilon(w)\,\exp\{w(\lambda+\rho)\}\sum_{w'\in W}P(w')e(-\rho-w'). \tag{2.30}$$

Comparing the left-hand side with the right-hand side in the above expression we obtain finally the Kostant multiplicity formula:

$$n_w(\lambda)=\sum_{w'\in W}\varepsilon(w')P(w'(\lambda+\rho)-(\rho+w)). \tag{2.31}$$

The obtained formula allows us to determine the density of states $n_w(\lambda)$. Provided that the function P is known explicitly, the obtained formula allows us to determine the factor $n_w(\lambda)$.

It is useful to rewrite these results in physical language. In particular, for any quantum mechanical system, the partition function Ξ can be written as

$$\Xi=\sum_n g_n\exp\{-\beta E_n\}\equiv tr(\exp(-\beta\hat{H})) \tag{2.32}$$

where $\hat{H}$ is the quantum Hamiltonian of the system, β the inverse temperature and g_n is the degeneracy factor. Clearly, using Eqs. (2.27) and (2.28), one can identify Ξ with $\chi(\lambda)$. Next we introduce the density of states $\rho(E)$ via

$$\rho(E)=\sum_n g_n\delta(E-E_n). \tag{2.33}$$

Comparison between Eqs. (2.31) and (2.33) suggests that the function P can be identified with the density of states. Using $\rho(E)$ the partition function Ξ can be written as the Laplace transform

$$\Xi(\beta)=\int_0^\infty \mathrm{d}E\rho(E)\exp\{-\beta E\}. \tag{2.34}$$

Clearly, Eq. (2.28) is just the discrete analogue of Eq. (2.34) so that it does have a statistical/quantum mechanical interpretation as partition function. From condensed matter physics it is known that all important statistical/quantum information is contained in the density of states. Its calculation is of primary interest in physics. Evidently, the same is true in the present case.

In the light of results just obtained, we can reinterpret some results from the Introduction. In particular, the right-hand side our Eq. (1.11), when compared with the right-hand side of Eq. (5.11) of AB paper [30, Part II], is not looking the same. We would like to explain that, nevertheless, these expressions are equivalent. For comparison, let us reproduce Eq. (5.11) of AB paper first. Actually, for this purpose it is more convenient to use the paper by Bott [35], (his Eq. (28)). In notations taken from this reference, Eq. (5.11) by AB is written now as follows:

$$trace\, T_{\rm g} = \sum_{\substack{w\in W\\ \alpha<0}} \left[\frac{\lambda}{\prod(1-\alpha)}\right]^{w}. \tag{2.35}$$

Comparing this result with the right-hand side of our Eq. (1.11) and taking into account Eq. (2.17), the combination λ^{w} in the numerator of Eq. (2.35) is the same thing as $\exp\{w\lambda\}$ in Eq. (2.27). As for the denominator, Bott uses the same Eq. (2.25) as we do so that it remains to demonstrate that

$$\left[\prod_{\alpha\in\Delta^{+}}(1-\exp(-\alpha))\right]^{w} = \exp(-w(\rho))\,\varepsilon(w)\hat{d}. \tag{2.36}$$

In view of Eq. (2.25), we need to demonstrate that

$$\left[\hat{d}\exp(-\rho)\right]^{w} = \exp(-w(\rho))\varepsilon(w)\hat{d}, \tag{2.37}$$

i.e. that $[\hat{d}]^{w} = \varepsilon(w)\hat{d}$. Looking at Eq. (2.23), this requires us to assume that $[\hat{d}]^{w} = w\hat{d}$. But, in view of Eqs. (2.17), (2.19) and (2.24), we conclude that this is indeed the case. This proves the fact that Eq. (2.35), that is Eq. (5.11) of Ref. [30], is indeed the same thing as the Weyl's character formula, Eq. (5.12) of Ref. [30], or, equivalently, Eq. (2.27) above. According to Kac [36, p. 174], the classical Weyl character formula, Eq. (2.27), is formally valid for both finite dimensional semisimple Lie algebras and infinite dimensional affine Kac–Moody algebras. This circumstance and the Proposition A.1 of Appendix play important motivating role for developments in our work.

Remark 2.2. Although our arguments thus far have been limited only to (comparison with) the d-dimensional hypercube, this deficiency is easily correctable with help of the concept of *zonotope* introduced in Section 2.1. Clearly, because of zonotope construction the obtained results remain correct for *any* centrally symmetric polytope $\mathcal{P}$. Thus, we have demonstrated that Eq. (1.11) has essentially the same meaning as the Weyl character formula.

Remark 2.3. In view of Eqs. (2.22) and (2.35) the denominator $\hat{d}$ in the Weyl character formula, Eq. (2.27), is actually a determinant. This means that the basic anti invariant $J(x)$ introduced in Eq. (2.19) is determinant. But the right-hand side of Eq. (2.27) is invariant. This is possible only if the numerator of the Weyl character formula is also a determinant. Hence, the Weyl character formula is essentially the ratio of determinants. Since this is surely the case, it implies that the Ruele zeta function, Eq. (2.13) is also essentially the Weyl character formula.[11] This, in turn, implies that our major result for the Veneziano partition function, Eq. (1.1), is essentially also the Weyl character formula in accord with Lerche et al. [4], where such conclusion was obtained differently.

The fact that Eq. (1.1) can be interpreted as the Weyl character formula should not be too surprising in view of the fact that the left-hand side of Eq. (1.1) denotes the Poincare polynomial which is group invariant. It remains to demonstrate that the torus action group introduced in Part II can be interpreted as the Weyl–Coxeter reflection group. This is done below, in Section 3. In the meantime, we have not exhausted all consequences of the results we have just obtained. In

[11] That this is the case for dynamical systems can be deduced, based on our arguments, from the monograph by Feres [37]. We shall not develop this line of thought in this paper having in mind different goals.

particular, if it is true that Eq. (1.11) is the Weyl character formula, then, taking into account Eq. (2.28), we conclude that Eq. (1.7) is the Kostant multiplicity formula for d-dimensional cube. The rigorous proof of this fact for the arbitrary convex polytope can be found in Refs. [33,38,39]. If this is so, then Eq. (1.2) is also the Kostant multiplicity formula providing the number of points inside the inflated (with inflation factor q) simplex $q\Delta_n$ (living in $\mathbf{Z}^n$ lattice) and at its boundaries. These observations allow us to develop symplectic methods for reconstruction of the Veneziano partition function to be discussed below in Section 4.

Remark 2.4. From the point of view of algebraic geometry of toric varieties [40–42], the Kostant multiplicity formula has yet another (topological) interpretation as the Euler characteristic χ of the *projective* toric variety associated with the polytope $\mathcal{P}$. This fact will be discussed in some detail below, in Sections 3.4 and 4. It motivates us to develop symplectic formulation of the Veneziano partition function in Section 4 and supersymmetric formulation discussed in Part II. Connections between the Weyl character formula and the Euler characteristic for projective algebraic varieties had been uncovered by Nielsen [43], already in 1976. His work is based on still earlier work by Iversen and Nielsen [44]. Below we shall argue that, actually, the idea of such connection can be traced back to still much earlier papers by Hopf [45], and Hopf and Samelson [46]. In Section 3.4, we provide the topological interpretation of the Kostant multiplicity formula in terms of χ based on ideas of Hopf and Samelson.

3. The torus action and the moment map

3.1. The torus action and the Weyl group

To avoid duplications, in writing this and following subsections we shall assume that our readers are familiar with our earlier work, Part II. Hence, in this part we only introduce terminology which is of immediate use. To begin, let us consider a polynomial

$$f(\mathbf{z}) = f(z_1, \ldots, z_n) = \sum_{\mathbf{i}} \lambda_{\mathbf{i}} \mathbf{z}^{\mathbf{i}} = \sum_{\mathbf{i}} \lambda_{i_1,\ldots,i_n} z_1^{i_1}, \ldots, z_n^{i_n}, \ (\lambda_{\mathbf{i}},\ z_m^{i_m} \in \mathbf{C}, 1 \le m \le n). \tag{3.1}$$

It belongs to the polynomial ring $A[\mathbf{z}]$ (essentially isomorphic to earlier introduced $A[\mathbf{X}]$) closed under ordinary addition and multiplication. Since now we are using complex numbers (instead of indeterminates as in Section 2.3) this allows us to introduce the following:

Definition 3.1. An *affine* algebraic variety $V \in \mathbf{C}^n$ is the set of zeros of the collection of polynomials from the ring $A[\mathbf{z}]$.

According to the famous Hilbert's Nullstellensatz a collection of such polynomials is *finite* and forms the set $I(\mathbf{z}) := \{f \in A[\mathbf{z}], f(\mathbf{z}) = 0\}$ of maximal ideals usually denoted *Spec* A[**z**].

Definition 3.2. The zero set of a *single* function belonging to $I(\mathbf{z})$ is called *algebraic hypersurface* so that the set $I(\mathbf{z})$ corresponds to the *intersection* of a finite number of hypersurfaces.

As in Part II, we need to consider the set of Laurent monomials of the type $\lambda \mathbf{z}^{\boldsymbol{\alpha}} \equiv \lambda z_1^{\alpha_1}, \ldots, z_n^{\alpha_n}$. We shall be particularly interested in the *monic* monomials for which $\lambda = 1$. Such monomials form a closed polynomial *subring* with respect to usual multiplication and addition. The crucial step forward is to assume that the exponent $\boldsymbol{\alpha} \in \mathbf{S}_\sigma$.[12] This allows us to define the following

[12] Although the monoid $\mathbf{S}_\sigma$ was defined in Part II, for reader's convenience it will be redefined below momentarily.

mapping

$$u_i := z^{a_i} \tag{3.2}$$

with a_i being one of the generators of the monoid $\mathbf{S}_\sigma$ and $z \in \mathbf{C}$. In order to define the monoid $\mathbf{S}_\sigma$ we still need to provide a couple of definitions. In particular, recall that *a semi-group S* that is a non-empty set with associative operation is called *monoid* if it is commutative, satisfies cancellation law (i.e. $s+x=t+x$ implies $s=t$ *for all* $s,t,x \in S$) and has zero element (i.e. $s+0=s, s\in S$). This allows us to make the following:

Definition 3.3. *A monoid S is finitely generated* if exist a set $a_1, \ldots, a_k \in S$, called *generators*, such that

$$S = Z_{\geq 0}a_1 + \cdots + Z_{\geq 0}a_k. \tag{3.3}$$

Taking into account this definition, it is clear that the monoid $\mathbf{S}_\sigma = \sigma \cap \mathbf{Z}^d$ for the rational polyhedral cone σ (e.g. read Part II) is *finitely generated*.

The mapping given by Eq. (3.2) provides an isomorphism between the *additive* group of exponents a_i and the *multiplicative* group of monic Laurent polynomials. Next, the function ϕ is considered to be *quasi homogenous* of degree d with exponents $l_1, \ldots, l_n$ if

$$\phi(\lambda^{l_1}x_1, \ldots, \lambda^{l_n}x_n) = \lambda^d \phi(x_1, \ldots, x_n), \tag{3.4}$$

provided that $\lambda \in \mathbf{C}^*$. Applying this result to $z^{\mathbf{a}} \equiv z_1^{a_1}, \ldots, z_n^{a_n}$ we obtain equation analogous to Eq. (3.3) for the polyhedral cone:

$$\sum_j (l_j)_i a_j = d_i. \tag{3.5}$$

Clearly, if the index i is numbering different monomials, then the sum d_i belongs to the monoid $\mathbf{S}_\sigma$. The same result can be achieved if instead we would consider the products of the type $u_1^{l_1}, \ldots, u_n^{l_n}$ and rescale all $z_i's$ by the same factor λ. Eq. (3.5) should be understood as a scalar product with $(l_j)_i$ living in the space *dual* to $a_j's$. Accordingly, the set of $(l_j)_i's$ can be considered as the set of generators for the dual cone $\sigma^\vee$. Next, in view of Eq. (3.2), let us consider the polynomials of the type

$$f(\mathbf{z}) = \sum_{\mathbf{a}\in\mathbf{S}_\sigma} \lambda_{\mathbf{a}} \mathbf{z}^{\mathbf{a}} = \sum_{\mathbf{l}} \lambda_{\mathbf{l}} \mathbf{u}^{\mathbf{l}}. \tag{3.6}$$

As before, they form a polynomial ring. The ideal for this ring is constructed based on the observation that for the fixed d_i and the assigned set of cone generators a_i there is more than one set of generators for the dual cone. This redundancy produces relations of the type

$$u_1^{l_1}, \ldots, u_k^{l_k} = u_1^{\tilde{l}_1}, \ldots, u_k^{\tilde{l}_k}. \tag{3.7}$$

If now we require $u_i \in \mathbf{C}_i$, then it is clear that the above equation belongs to the ideal $I(\mathbf{z})$ of the above polynomial ring and that Eq. (3.7) represents the hypersurface in accord with Definitions 3.1 and 3.2. As before, the ideal $I(\mathbf{z})$ represents the intersection of these hypersurfaces thus forming the affine *toric* variety $X_{\sigma^\vee}$. The generators $\{u_1, \ldots, u_k\} \in \mathbf{C}^k$ are coordinates for $X_{\sigma^\vee}$. They represent the same point in $X_{\sigma^\vee}$ if and only if $\mathbf{u}^{\mathbf{l}} = \mathbf{u}^{\tilde{\mathbf{l}}}$. Thus formed toric variety corresponds to just one (dual) cone. A *complex algebraic torus T* is defined by the rule $T := (\mathbf{C}\backslash 0)^n =: (\mathbf{C}^*)^n$. It acts on the affine toric variety X_Σ (made out of pieces $X_{\sigma^\vee}$ with help of a gluing map) according to the prescription: $T \times X_\Sigma \to X_\Sigma$, provided that at each affine variety corresponding to the dual cone $\sigma^\vee$ its action is given by

$$T \times X_{\sigma^\vee} \to X_{\sigma^\vee}, \quad (t, x) \mapsto tx := (t^{a_1}x_1, \ldots, t^{a_k}x_k). \tag{3.8}$$

To proceed, we us replace temporarily T by the group G acting (multiplicatively) on the set X via the rule: $G \times X \to X$, i.e. $(g, x) \to gx$, provided that for all $g, h \in G$, $g(hx) = (gh)x$ and $ex = x$ for some unit element e of G.

Definition 3.4. The subset $Gx := \{gx \mid g \in X\}$ of X is called the *orbit* of x. The subgroup $H := \{gx = x \mid g \in X\}$ of G that fixes x is the *isotropy* group. There could be more than one fixed point for the equation $gx = x$. All of them are conjugate to each other.

Definition 3.5. A *homogenous* space for G is the subspace of X on which G acts without fixed points.

The major step forward can be made by introducing the concept of an *algebraic* group [47].

Definition 3.6. A *linear algebraic group* G is (a) an affine algebraic variety and (b) a group in the sense given above, i.e.

$$\mu : G \times G \to G; \mu(x, y) = xy \tag{3.9a}$$

$$i : G \to G; \iota(x) = x^{-1}. \tag{3.9b}$$

Remark 3.7. It can be shown, Ref. [48, p. 150], that G as a linear algebraic group is isomorphic to a closed subgroup of $GL_n(K)$ for some $n \geq 1$ and any closed number field K such as **C** or p-adic. This fact plays the central role in whole development presented below.

Consider therefore an action of G on $f(\mathbf{z})$ defined by Eq. (3.6). Following Stanley [13], it can be defined as $M \circ f(\mathbf{z}) = f(M\mathbf{z})$ for some matrix M such that $M \in G$. In order for this definition to be compatible with earlier made, Eq. (3.8), we have to assume that the torus T acts diagonally on the vector space spanned by $x_1, \ldots, x_n$. This means that the isotropy group of the torus is defined by the set of the following equations

$$t^{a_i}x_i = x_i. \tag{3.10}$$

Apart from trivial solutions: $x_i = 0$ and $x_i = \infty$, there are nontrivial solutions: $t^{a_i} = 1 \forall x_i$. For integer $a_i^\prime s$ this are cyclotomic equations whose $a_i - 1$ solutions all lie on the circle (e.g. see Part I, Section 3.1). This result is easy to understand since the algebraic torus T has the topological torus as a deformation retract while the topological torus is just a Cartesian product of circles.

Next, we notice that Eq. (3.8) still makes sense if some of t-factors are replaced by 1s. This means that one should take into account situations when one, two, etc., t-factors in Eq. (3.8) are replaced by 1s and account for all permutations involving such cases. This observation leads to the torus actions on toric subvarieties. It is important that different orbits belong to different subvarieties which do not overlap. Thus, by design, X_Σ is the disjoint union of *finite* number of orbits identified with the subvarieties of X_Σ. Under such circumstances the vector $(x_1, \ldots, x_k)$ forms a basis of k-dimensional vector space V so that the vector $(x_1, \ldots, x_i)$, $i \leq k$, forms a basis of the subspace V_i. This allows us to introduce a complete flag f_0 of subspaces in V via

$$f_0 : 0 = V_0 \subset V_1 \subset \ldots \subset V_k = V. \tag{3.11}$$

Consider now an action of G on f_0. Taking into account Remark 3.7., we recognize that effectively $G = GL_n(K)$. The matrix representation of this group possess remarkable properties. These are summarized in the following definitions.

Definition 3.8. Given that the set $GL_n(K) = \{x \in M_n(K) \mid \det x \neq 0\}$ with $M_n(K)$ being $n \times n$ matrix with entries $x_{i,j} \in K$ forms a general linear group, the matrix $x \in M_n(K)$ is (a) *semisimple* ($x = x_s$), if it is diagonalizable, that is $\exists g \in GL_n(K)$ such that gxg^{-1} is a diagonal matrix; (b) *nilpotent* ($x = x_n$) if $x^m = 0$, that is for some positive integer m all eigenvalues of the matrix x^m are zero; (c) *unipotent* ($x = x_u$), if $x - 1_n$ is nilpotent, i.e. x is the matrix whose only eigenvalues are 1's.

Just like with the odd and even numbers the above matrices, *if they exist*, form closed disjoint subsets of $GL_n(K)$, e.g. all $x, y \in M_n(K)$ commute; if x, y are semisimple so is their sum and the product, etc. Most important for us is the following fact:

Proposition 3.9. *Let $x \in GL_n(K)$. Then $\exists x_u$ and x_s such that $x = x_s x_u = x_u x_s$. Both x_s and x_u are determined by the above conditions uniquely.*

The proof can be found in Ref. [49, p. 96]. This proposition is in fact a corollary of the Lie–Kolchin theorem which is of major importance for us. To formulate this theorem we need to introduce yet another couple of definitions. In particular, if A and B are closed (finite) subgroups of the algebraic group G one can construct the group (A, B) made of commutators $xyx^{-1}y^{-1}$, $x \in A$, $y \in B$. With help of such commutators the following definition can be made.

Definition 3.10. The group G is *solvable* if its derived series terminates in the unit element e. The derived series is being defined inductively by $\mathcal{D}^{(0)}G = G$, $\mathcal{D}^{(i+1)}G = (\mathcal{D}^{(i)}G, \mathcal{D}^{(i)}G)$, $i \geq 0$.

Such a definition implies that the algebraic group G is solvable if and only if there exists a chain $G = G^{(0)} \supset G^{(1)} \supset \cdots G^{(n)} = e$ for which $(G^{(i)}, G^{(i)}) \subset G^{i+1}$ $(0 \leq i \leq n)$, Ref. [49, p. 111]. Finally,

Definition 3.11. The group is called *nilpotent* if $\mathcal{E}^{(n)}G = e$ for some n, where $\mathcal{E}^{(0)} = G, \mathcal{E}^{(i+1)} = (G, \mathcal{E}^{(i)}G)$.

Such a group is represented by the nilpotent matrices. Based on this definition, it is possible to prove that every nilpotent group is solvable [40, p. 112]. These results lead us to the Lie–Kolchin theorem of major importance:

Theorem 3.12 (Lie and Kolchin [49, p. 113]). *Let G be connected solvable algebraic group acting on a projective variety X. Then G has a fixed point in X.*

In view of Remark 3.7, we know that such G is a subgroup of $GL_n(K)$. Moreover, $GL_n(K)$ has at least another subgroup, called *semisimple*, for which Theorem 3.12 does not hold. In this case we have the following definition.

Definition 3.13. The group G is *semisimple* if it has no closed connected commutative normal subgroups other than e.

Such a group is represented by the semisimple, i.e. diagonal (or torus), matrices while the members of the unipotent group are represented by the upper triangular matrices with all diagonal entries being equal to 1. In view of Theorem 3.12, the unipotent group is also solvable and, accordingly, there must be an element B of such a group fixing the flag f_0 defined by Eq. (3.11), i.e.

$Bf_0 = f_0$. Let now $g \in GL_n(K)$. Then, naturally, $gf_0 = f$ where $f \neq f_0$. From here we obtain, $f_0 = g^{-1}f$. Next, we obtain as well, $Bg^{-1}f = g^{-1}f$ and, finally, $gBg^{-1}f = f$. Based on these results, it follows that $gBg^{-1} = \tilde{B}$ is also an element of $GL_n(K)$ fixing the flag *f*, etc. This means that all such elements are conjugate to each other and form the *Borel subgroup*. We shall denote all elements of this sort by *B*. These are made of upper triangular matrices belonging to $GL_n(K)$. Surely, such matrices satisfy Proposition 3.9. The quotient group G/B will act transitively on *X*. Since this quotient is also a linear algebraic group, it is as well a projective variety called the *flag variety*, Ref. [48, p. 176].

Remark 3.14. The flag variety is directly connected with the *Schubert variety*, Ref. [50, p. 124]. The Schubert varieties were considered earlier, in our work, Ref. [51], in connection with the exact combinatorial solution of the Kontsevich–Witten (K–W) model. Hence, the above remark naturally leads us to the combinatorial approach to problems we are discussing in this part of our work and in Part IV. Additional details on connections with K–W model will become apparent in Section 4.

By now it should be clear that the group *G* is made out of at least two subgroups: *B*, just described, and *N*. The *maximal torus T* subgroup of *G* can be defined now as $T = B \cap N$. This fact allows us to define the Weyl group: $W = N/T$. Although this group has the same name as that discussed in the Appendix, its true meaning in the present context requires some explanations. They will be provided below.

This is done in several steps. First, using results of appendix we notice that the "true" Weyl group is made of reflections, i.e. involutions of order 2. Following Tits [32], we introduce a quadruple (*G*, *B*, *N*, *S*) (the Tits system) where *S* is the subgroup of *W* made of elements such that $S = S^{-1}$ and $1 \notin S$. Such a subroup always exists for the compact Lie groups considered as symmetric spaces. Then, it can be shown that $G = BWB$ (*Bruhat decomposition*) and, moreover, that the Tits system is isomorphic to the Coxeter system, i.e. to the Coxeter reflection group. The full proof can be found in the monograph by Bourbaki [32], Chr. 6, paragraph 2.4.

Second, since $W = N/T$, it is of interest to see the connection (if any) between *W* and the quotient $G/B = BWB/B = [B(N/T)B]/B$. In view of the fact that $T = B \cap N$, suppose that *N* commutes with *B*. Then we would have $G/B \simeq (N/T)B$ and, since *B* fixes the flag *f*, we are left with the action of *N* on the flag. In view of the rule: $M \circ f(\mathbf{z}) = f(M\mathbf{z})$, and noticing that the diagonal matrix *T* (the centralizer) can be chosen as a reference (identity) transformation, we conclude that the commuting matrix *N* (the normalizer) should permute t^{a_i}. Consider an application of this rule to the monomial $\mathbf{u}^{\mathbf{l}} = u_1^{l_1}, \ldots, u_n^{l_n} \equiv z_1^{l_1 a_1}, \ldots, z_n^{l_{n1} a_n}$. For such a map the character $c(t)$ is given by

$$c(t) = t^{\langle \mathbf{l} \cdot \mathbf{a} \rangle}, \tag{3.12}$$

where, in accord with Eq. (3.5), $\langle \mathbf{l} \cdot \mathbf{a} \rangle = \sum_i l_i a_i$ with both l_i and a_i being some integers. Following Ref. [52], let us consider the limit $t \to 0$ in the above expression. Clearly, we obtain:

$$c(t) = \begin{cases} 1 & \text{if } \langle \mathbf{l} \cdot \mathbf{a} \rangle = 0 \\ 0 & \text{if } \langle \mathbf{l} \cdot \mathbf{a} \rangle \neq 0 \end{cases}. \tag{3.13}$$

Evidently, the equation $\langle \mathbf{l} \cdot \mathbf{a} \rangle = 0$ describes a hyperplane or, better, a set of hyperplanes for a given vector $\mathbf{a}$. In view of Eq. (3.5), such a set forms at least one polyhedral cone (or chamber in the terminology of Appendix). These results can be complicated a little bit by introducing a subset $I \subset \{1, \ldots, n\}$ such that, say, only those $l_i's$ which belong to this subset satisfy $\langle \mathbf{l} \cdot \mathbf{a} \rangle = 0$.

Naturally, one obtains the one-to-one correspondence between such subsets and earlier defined flags. Clearly, the set of such constructed monomials forms an invariant of the torus group action as discussed in Part II. It remains to demonstrate that the Weyl group $W = N/T$ permutes a_i's thus forming an orbit transitively "visiting" different hyperplanes. This will be demonstrated momentarily. Before doing this, we would like to change the rules of the game slightly.[13] To this purpose, we would like to replace the limiting $t \to 0$ procedure by the procedure requiring $t \to \xi$ with ξ being the nontrivial n-th root of unity. After such a replacement we are entering the domain of the pseudo-reflection groups discussed in Part II. Thus, replacing t by ξ causes us to change the rule, Eq. (3.13), as follows:

$$c(\xi) = \begin{cases} 1 & \text{if } \langle \mathbf{l} \cdot \mathbf{a} \rangle = 0 \bmod n \\ 0 & \text{if } \langle \mathbf{l} \cdot \mathbf{a} \rangle \neq 0 \end{cases}. \tag{3.14}$$

At this point it is appropriate to recall Eq. (I, 3.11a). In view of this equation, we shall call the equation $\langle \mathbf{l} \cdot \mathbf{a} \rangle = n$ as the *Veneziano condition* while the *Kac–Moody–Bloch–Bragg* (*K–M–B–B*) *condition*, Eq. (I, 3.22), can be written now as $\langle \mathbf{l} \cdot \mathbf{a} \rangle = 0 \bmod n$.

The results of Appendix indicate that the first option (the Veneziano condition) is characteristic for the standard Weyl–Coxeter (pseudo) reflection groups while the second is characteristic for the affine Weyl–Coxeter groups thus leading to the Kac–Moody affine Lie algebras as discussed in Part II.

At this moment we are ready to demonstrate that $W = N/T$ is indeed the Weyl reflection group. Even though the full proof can be found, for example, in the monograph by Bourbaki [32], still it is instructive to provide qualitative arguments exhibiting the essence of the proof (different from that given by Bourbaki who use the Tits system).

Let us begin with an assembly of $(d+1) \times (d+1)$ matrices with complex coefficients. They belong to the group $GL_{d+1}(\mathbf{C})$. Consider a subset of all diagonal matrices and, having in mind physical applications, let us assume that the diagonal entries are made of n-th roots of unity ξ. Taking into account the results on pseudo-reflection groups as discussed in Appendix, each diagonal entry can be represented by ξ^k with $1 \le k \le n-1$ so that there are $(n-1)^{d+1}$ different diagonal matrices-all commuting with each other. Among these commuting matrices we would like to single out those which have all $\xi^{k'} s$ the same. Evidently, there are $n-1$ of them. They are effectively the unit matrices and they are forming the centralizer of W. The rest belongs to the normalizer.[14] The number $(n-1)^{d+1}/(n-1) = (n-1)^d$ was obtained earlier, e.g. see the discussion which follows Eq. (1.7) (and replace $2m$ by n) and the discussion which follows this equation. This is not just a mere coincidence. In the next section we shall provide some refinements of this result motivated by physical considerations. It should be clear already that we are discussing only the simplest possibility (of cubic symmetry) for the sake of illustration of general principles. Clearly, the zonotope construction, introduced earlier allows us to transfer our reasoning to more general cases.

Next, let us consider just one of the diagonal matrices $\tilde{T}$ whose entries are all different and are made of powers of ξ. Let $g \in GL_{d+1}(\mathbf{C})$ and consider an automorphism: $\mathcal{F}(\tilde{T}) := g\tilde{T}g^{-1}$. Along with it, we would like to consider an orbit $O(\tilde{T}) := g\tilde{T}C$ where C is any of the diagonal matrices

[13] Such change of rules is consistent with arguments by Kirillov to be discussed in the next subsection.

[14] As with Eq. (3.12), one can complicate matters by considering matrices which have several diagonal entries which are the same. Then, as before, one should consider the flag system where in each subsystem the entries are all different. The arguments applied to such subsystems will proceed the same way as in the main text.

belonging to earlier discussed centralizer.[15] Clearly, $O(\tilde{T}) = g\tilde{T}g^{-1}gC = \mathcal{F}(\tilde{T})gC = \mathcal{F}(\tilde{T})C$. Denote now $\tilde{T} = \tilde{T}_1$ and consider another matrix $\tilde{T}_2$ belonging to the same set and suppose that there is such matrix g_{12} that $\tilde{T}_2 C = \mathcal{F}(\tilde{T})C$. If such a matrix exists, it should belong to the normalizer and, naturally, the same arguments can be used to $\tilde{T}_3$, etc. Hence, the following conclusions can be drawn. If we had started with some element $\tilde{T}_1$ of the maximal torus, the orbit of this element will return back and intersect the maximal torus in *finite* number of points (in our case the number of points is exactly $(n-1)^d$). By analogy with the theory of dynamical systems, we can consider these intersection points of the orbit $O(\tilde{T})$ with the T-plane as the Poincare$'$ crossections. Hence, as it is done in the case of dynamical systems (e.g. see Section 2.2), we have to study the transition map between these crossections. The orbit associated with such a map is precisely the orbit of the Weyl group W. It acts on these points transitively [49, p. 147], as required. Provided that the set of fixed point exists, such arguments justify the dynamical interpretation of the Weyl's character formula presented in Section 2.2. The fact that such fixed point set does exist is guaranteed by the Theorem 10.6. by Borel [47]. Its proof relies heavily on the Lie–Kolchin theorem (our Theorem 3.12).

3.2. Coadjoint orbits

Thus far we were working with the Lie groups. To move forward, we need to use the Lie algebras associated with these groups. In what follows, we expect our readers familiarity with basic relevant facts about the Lie groups and Lie algebras which can be found in the books by Serre [53], Humphreys [54] and Kac [36]. First, we notice that the Lie algebras matrices h_i associated with the Lie group maximal tori T_i (that is with all diagonal matrices considered earlier) are commuting with each other thus forming the Cartan subalgebra, i.e.

$$[h_i, h_j] = 0. \tag{3.15}$$

The matrices belonging to the normalizer are made of two types x_i and y_i corresponding to the root systems Δ^+ and Δ^- defined in Appendix. The fixed point analysis described at the end of previous section is translated into the following set of commutators:

$$[x_i, y_j] = \begin{cases} h_i, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}, \tag{3.16a}$$

$$[h_i, x_j] = \langle \alpha_{i^\vee}, \alpha_j \rangle x_j, \tag{3.16b}$$

$$[h_i, y_j] = -\langle \alpha_{i^\vee}, \alpha_j \rangle y_j, \tag{3.16c}$$

$i = 1, \ldots, n$. To insure that the matrices (operators) $x_i's$ and $y_i's$ are nilpotent (that is their Lie group ancestors belong to the Borel subgroup B) one must impose two additional constraints. According to Serre [53] these are:

$$(\mathrm{ad}x_i)^{-\langle \alpha_{i^\vee}, \alpha_j \rangle + 1}(x_j) = 0, \quad i \neq j \tag{3.16d}$$

$$(\mathrm{ad}y_i)^{-\langle \alpha_{i^\vee}, \alpha_j \rangle + 1}(y_j) = 0, \quad i \neq j. \tag{3.16e}$$

[15] Presence of C factor underscores the fact that we are considering the orbit of the factorgroup $W = N/T$.

where $\mathrm{ad}_X\, Y = [X, Y]$. From the book by Kac [36] one finds that *exactly the same* relations characterize the Kac–Moody affine Lie algebra. This fact is in accord with general results presented earlier in this work and in Part II and is of major importance for development of our formalism. In particular, for the purposes of this development it is important to realize that for each i Eqs. (3.16a)–(3.16c) can be brought to form (upon rescaling) coinciding with the Lie algebra $sl_2(\mathbf{C})$[16] and, if we replace $\mathbf{C}$ with any closed number field $\mathbf{F}$, then all semisimple Lie algebras are made of copies of $sl_2(\mathbf{F})$ [54, p. 25]. The Lie algebra $sl_2(\mathbf{C})$ is isomorphic to the algebra of operators acting on differential forms living on the Hodge-type complex manifolds [55]. This observation was absolutely essential for development of physical applications in Part II.

Connections with Hodge theory can be also established through the method of coadjoint orbits. We would like to discuss this method now. We begin by considering an orbit in the Lie group. It is given by the Ad operator, i.e. $O(X) = \mathrm{Ad}_g X = gXg^{-1}$ where $g \in G$ and $X \in \mathfrak{g}$ with G being the Lie group and $\mathfrak{g}$ its Lie algebra. For compact groups globally and for noncompact locally every group element g can be represented via the exponential, e.g. $g(t) = \exp(tX_g)$, with $X_g \in \mathfrak{g}$. Accordingly, for the orbit we can write $O(X) \equiv X(t) = \exp(tX_g)X\exp(-tX_g)$. Since the Lie group is a manifold $\mathcal{M}$, the Lie algebra forms the tangent bundle of the vector fields at given point of $\mathcal{M}$. In particular, the tangent vector to the orbit $X(t)$ is determined, as usual, by $TO(X) = \frac{\mathrm{d}}{\mathrm{d}t}X(t)_{t=0} = [X_g, X] = ad_{X_g}X$. Now we have to take into account that, actually, our orbit is made for a vector X coming from the torus, i.e. $T = \exp(tX)$. This means that when we consider the commutator $[X_g, X]$ it will be zero for $X_{g_i} = h_i$ and nonzero otherwise. Consider next the Killing form $\kappa(x, y)$ for two elements x and y of the Lie algebra:

$$\kappa(x, y) = tr(adx\, ady). \tag{3.17}$$

From this definition it follows that

$$\kappa([x, y], z) = \kappa(x, [y, z]). \tag{3.18}$$

The roots of the Weyl group can be rewritten in terms of the Killing form [54]. Its purpose is to define the scalar multiplication between vectors belonging to the Lie algebra and, as such, it allows one to determine the notion of orthogonality between these vectors. In particular, if we choose $x \to X$ and $y, z \in h_i$, then it is clear that the vector tangential to the orbit $O(X)$ is going to be orthogonal to the subspace spanned by the Cartan subalgebra. This result can be reinterpreted from the point of view of symplectic geometry due to work of Kirillov [57]. To this purpose we would like to rewrite Eq. (3.18) in the equivalent form, i.e.

$$\kappa(x, [y, z]) = \kappa(x, \mathrm{ad}_y z) = \kappa(\mathrm{ad}_x^* y, z) \tag{3.19}$$

where in the case of compact Lie group, $\mathrm{ad}_x^* y$ actually coincides with $\mathrm{ad}_x y$. The reason for introducing the asterisk $(*)$ lies in the following chain of arguments. In Eq. (A.1) of Appendix (and in Eq. (3.5)) we introduced vectors from the dual space. Such a construction is possible as long as the scalar multiplication is defined. Hence, for the orbit $\mathrm{Ad}_g X$ there must be a vector ξ in the dual space $\mathfrak{g}^*$ such that equation

$$\kappa(\xi, \mathrm{Ad}_g X) = \kappa(\mathrm{Ad}_g^* \xi, X) \tag{3.20}$$

[16] This fact is known as Jacobson–Morozov theorem [56].

defines the *coadjoint orbit* $O^*(\xi) = \mathrm{Ad}^*_g\xi$. Accordingly, for such an orbit there is also the tangent vector $TO^*(\xi) = \mathrm{ad}^*_{\mathsf{g}}\xi$ to the orbit and, clearly, we have $\kappa(\xi, ad_{X_g}X) = \kappa(ad^*_{\mathsf{g}}\xi, X)$. In the case if we are dealing with the flag space, the family of coadjoint orbits will represent the flag space structure as well. Next, let $x \in \mathsf{g}^*$ and $\xi_1, \xi_2 \in TO^*(x)$. Then consider the properties of the (symplectic) form $\omega_x(\xi_1, \xi_2)$ to be determined explicitly momentarily. For this purpose we need to introduce some notations, e.g. $ad^*_{\mathsf{g}}x = f(x, \mathsf{g})$, etc. so that for g_1 and $\mathsf{g}_2 \in \mathsf{g}$ one has $\xi_i = f(x, \mathsf{g}_i)$, $i = 1, 2$. Then, one can claim that for the compact Lie group and the associated with it Lie algebra $\omega_x(\xi_1, \xi_2) = \kappa(x, [\mathsf{g}_1, \mathsf{g}_2])$. Indeed, using Eq. (3.18) we obtain: $\kappa(x, [\mathsf{g}_1, \mathsf{g}_2]) = \kappa(\xi_1, \mathsf{g}_2) = -\kappa(x, [\mathsf{g}_2, \mathsf{g}_1]) = -\kappa(\xi_2, \mathsf{g}_1)$. Thus constructed form defines the symplectic structure on the coadjoint orbit $O^*(x)$ since it is closed, skew-symmetric, nondegenerate and is effectively independent of the choice of g_1 and g_2. The proofs can be found in the literature [58]. The obtained symplectic manifold $\mathcal{M}_x$ is the quotient g/g_h with g_h being made of vectors of the Cartan subalgebra so that for such vectors, by construction, $\omega_x(\xi_1, \xi_2) = 0$. From the point of view of symplectic geometry, the points for which $\omega_x(\xi_1, \xi_2) = 0$ correspond to the critical points for the velocity vector field on the manifold $\mathcal{M}_x$. I.e. these are the points at which the velocity field vanishes. They are in one-to one correspondence with the fixed points of the orbit $O(X)$. This fact allows us to use the Poincare′–Hopf index theorem (earlier used in our works on dynamics of $2 + 1$ gravity [59]) in order to obtain the Euler characteristic χ for such manifold as the sum of indices of vector fields existing on $\mathcal{M}_x$. We shall provide more details related to this observation below in Section 3.4.

To complete the above discussion, following Atiyah [60], we notice that every nonsingular algebraic variety in projective space is symplectic. The symplectic (Kähler) structure is inherited from that in the projective space. The complex Kähler structure for the symplectic (Kirillov) manifold is actually of the Hodge-type. This comes from the following observations. First, since we have used the Killing form to determine the Kirillov symplectic form ω_x and since the same Killing form is used for the Weyl reflection groups [58], the induced unitary one-dimensional representation of the torus subgroup of $GL_n(\mathbf{C})$ is obtained according to Kirillov [57] by simply replacing t by the root of unity in Eq. (3.12). This is permissible if and only if the integral of two-form $\int_\gamma \omega_x$ taken over any two-dimensional cycle γ on the coadjoint orbit $O^*(x)$ is an integer. But this is exactly the condition which makes the Kähler complex structure that of the Hodge type [55].

3.3. *Construction of the moment map using methods of linear programming*

In this subsection we are not employing the definition of the moment mapping used in symplectic geometry [61].[17] Instead, we shall rely heavily on works by Atiyah [60,62] with only slightest refinement coming from noticed connections with the linear programming not mentioned in his papers and in literature on symplectic geometry. In our opinion, such a connection is helpful for better physical understanding of mathematical methods discussed in this paper which potentially may be useful for applications in other disciplines.

Using Definition 1.1. of Section 1. We call the subset of $\mathbf{R}^n$ a polyhedron $\mathcal{P}$ if there exist $m \times n$ matrix A (with $m < n$) and a vector $b \in \mathbf{R}^m$ such that according to Eq. (1.8) we have

$$\mathcal{P} = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{Ax} \leq \mathbf{b}\}.$$

[17] Evidently, we are using the same thing anyway.

Since each component of the inequalities $\mathbf{Ax} \leq \mathbf{b}$ determines the half space while the equality $\mathbf{Ax} = \mathbf{b}$-the underlying hyperplane, the polyhedron is an intersection of finitely many halfspaces. The problem of linear programming can be formulated as follows [63]: for the linear functional $\tilde{\mathcal{H}}[\mathbf{x}] = \mathbf{c}^{\mathrm{T}} \cdot \mathbf{x}$ find $max\tilde{\mathcal{H}}[\mathbf{x}]$ on $\mathcal{P}$ provided that the vector $\mathbf{c}$ is assigned. It should be noted that this problem is just one of many related problems. It was selected only because of its immediate relevance. Its relevance comes from the fact that the extremum of $\tilde{\mathcal{H}}[\mathbf{x}]$ is achieved at least at one of the vertices of $\mathcal{P}$. The proof of this we omit since it can be found in any standard textbook on linear programming, e.g. see [64] and references therein. This result does not require the polyhedron to be centrally symmetric. Only convexity of the polyhedron is of importance. This is physically plausible since, for instance, reflexive polyhedra discussed in Section 1 in connection with mirror symmetry do not require such central symmetry as can be seen from two-dimensional examples presented in Ref. [65, p.100].

To connect this optimization problem with results of our paper we constrain $\mathbf{x}$ variables to integers, i.e. to $\mathbf{Z}^n$. Such a restriction is known in literature as *integer linear programming*. In our case, it is equivalent to considering symplectic manifolds of Hodge-type (e.g. read page 11 of Atiyah's paper, Ref. [60]). Hence, existence of mirror symmetry as well as the method of coadjoint orbits both require the underlying symplectic manifolds to be of Hodge-type. This has a deep physical reason which will become clear when we shall discuss the Khovanskii–Pukhlikov correspondence in the next section.

As a warm up exercise, following Fulton [40], let us consider a deformation retract of complex projective space $\mathbf{CP}^n$ which is the simplest possible toric variety [40,41].[18] Such a retraction is achieved by using the map:

$$\tau : \mathbf{CP}^n \to \mathbf{P}^n_{\geq} = \mathbf{R}^{n+1}_{\geq} \setminus \{0\}/\mathbf{R}^+$$

explicitly given by

$$\tau : (z_0, \ldots, z_n) \mapsto \frac{1}{\sum_i |z_i|}(|z_0|, \ldots, |z_n|) = (t_0, \ldots, t_n), \quad t_i \geq 0. \tag{3.21}$$

The map τ by design is onto the standard n-simplex: $t_i \geq 0$, $t_0 + \ldots + t_n = 1$. To bring physics to this discussion, let us consider the Hamiltonian for the harmonic oscillator. In the appropriate system of units we can write it as $\mathcal{H} = m(p^2 + q^2)$. More generally, for finite set of oscillators, i.e. for the "truncated" bosonic string, we have: $\mathcal{H}[\mathbf{z}] = \sum_i m_i |z_i|^2$, where, following Atiyah [60], we introduced the complex z_j variables via $z_j = p_j + iq_j$. Let now such a Hamiltonian system (the truncated string) possess the finite fixed energy $\mathcal{E}$. Then we obtain:

$$\mathcal{H}[\mathbf{z}] = \sum_{i=0}^{n} m_i |z_i|^2 = \mathcal{E}. \tag{3.22}$$

It is not difficult to realize that the above equation actually represents the $\mathbf{CP}^n$ since the points z_j can be identified with the points $e^{i\theta} z_j$ in Eq. (3.22) while keeping the above expression form-invariant. In such a case one is saying that the *reduced* phase space for this model is $\mathbf{CP}^n$ as discussed already in Section 7.6.3 of Part II. We can map such a model of $\mathbf{CP}^n$ back into the simplex using the map τ. Since $\mathbf{CP}^n$ is the simplest toric variety [40,41], if we let z_j to "live" in such a variety it will be affected by the torus action as discussed earlier in this section. This means that, in general, the masses in Eq. (3.22) may change and, accordingly, the energy. Only if

[18] Although such a construction was introduced in Part II, we write it down explicitly anyway for the sake of uninterrupted reading.

we constrain the torus action to the simplex (or, more generally, to the polyhedron as described by Fulton [40]), will the energy be conserved. Evidently, such a constraint is compatible with the original idea of identification of points z_j with $e^{i\theta}z_j$. The fixed points of such defined torus action are roots of unity according to Eq. (3.10). In general, the existence of at least one fixed point is guaranteed for the linear algebraic group by Theorem 10.6, Ref. [47]. With such defined torus action, $|z_i|^2$ is just some *positive* number, say, x_i. *The essence of the moment map lies exactly in such identification*.[19] Hence, we obtain the following image of the moment map:

$$\tilde{\mathcal{H}}[\mathbf{x}] = \sum_{i=0}^{n} m_i x_i, \tag{3.23}$$

where we have removed the energy constraint for a moment thus making $\tilde{\mathcal{H}}[\mathbf{x}]$ to coincide with earlier defined linear functional to be optimized. Now we have to find a convex polyhedron on which such a functional is going to be optimized. Thanks to works by Atiyah [60,62] and Guillemin and Sternberg [66], this task is completed already. Naturally, the vertices of such a polyhedron are the critical points of the moment map. Then, the theorem of linear programing stated earlier guarantees that $\tilde{\mathcal{H}}[\mathbf{x}]$ achieves its maximum at least at some of its vertices. Delzant [67] had demonstrated that this is the case without use of linear programming language.

It is helpful to illustrate the essence of above arguments by employing simple but important example originally discussed by Frankel [68]. Consider a two sphere S^2 of unit radius, i.e. $x^2 + y^2 + z^2 = 1$, and parametrize this sphere using coordinates $x = \sqrt{1-z^2}\cos\phi$, $y = \sqrt{1-z^2}\sin\phi$, $z = z$. In Section 4, we shall demonstrate that the Hamiltonian for the free particle “living” on such a sphere is given by $\mathcal{H}[z] = m(1-z)$ so that equations of motion produce the circles of latitude. These circles become (critical) points of equilibria at the north and south pole of the sphere, i.e. for $z = \pm 1$. Evidently, these are the fixed points of the torus action. Under such circumstances our polyhedron is the segment $[-1, 1]$ and its vertices are located at ± 1 (to be compared with discussion in Section 1). The image of the moment map $\mathcal{H}[x] = m(1-x)$ acquires its maximum at $x = 1$ and the value $x = 1$ corresponds to *two* polyhedral vertices located at 1 and -1, respectively. This doubling feature was noticed and discussed in detail by Delzant [67] whose work contains all needed proofs. These can be considered as elaborations on much earlier results by Frankel [68]. The circles on the sphere are representing the torus action (e.g. read the discussion following Eq. (3.22)) so that dimension of the circle is half of that of the sphere. This happens to be a general trend: the dimension of the Cartan subalgebra (more accurately, the normalizer of the maximal torus) is half of the dimension of the symplectic manifold $\mathcal{M}$ [39,67]. Incidentally, in the next subsection we shall see that the integral of the Kirillov–Kostant symplectic two-form ω_x over S^2 is equal to 2 so that the complex structure on the sphere is that of the Hodge type as required. Also, the symplectic two-form $\omega_x = 0$ at two critical points. Generalization of this example to the multiparticle case will be discussed below and in Section 4.

The results discussed thus far although establish connection between the singularities of symplectic manifolds and polyhedra do not allow us to discuss the fine details distinguishing between different polyhedra. Fortunately, this has been to a large degree accomplished in Refs. [58,69]. Such a task is equivalent to classification of all finite dimensional exactly integrable systems in accord with the Lie groups and Lie algebras associated with them.

[19] More accurate definition is given in Section 4.

3.4. Calculation of the Euler characteristic

Using results just presented we are ready to calculate the Euler characteristic of the projective algebraic variety following ideas by Hopf [45] and Hopf and Samelson [46]. To begin, we notice that in the case of vector fields on S^2 discussed in the previous subsection there are two fixed points. The Poincare$'$-Hopf fixed point theorem (extensively used in our earlier work on $2+1$ gravity, Ref. [59]) tells us that χ is the sum of indices of the vector (or line) fields foliating manifold $\mathcal{M}$. In our case, the index of each critical point is known to be 1 so that $\chi = 2$ as required.[20] In the case of S^2 the Darboux coordinates can be chosen as ϕ and z with $0 \leq \phi < 2\pi$ and $z \in [-1, 1]$. The volume form Ω is $\mathrm{d}\phi \wedge \mathrm{d}z$ so that the phase space is effectively the product $R \times S^1$. We would like to construct now a dynamical system whose Darboux coordinates are $\{t_1, \ldots, t_n; \phi_1, \ldots, \phi_n\}$. If in the case of S^2 the z coordinate varied in the segment $[-1, 1]$, now we shall assume that the point $\mathbf{t} = \{t_1, \ldots, t_k\}$ can vary inside some polytope $\mathcal{P} \subset \mathbf{R}^k$ including its boundaries. For our purposes, in view of Eq. (3.22), it is sufficient to consider only some simplex Δ_k living in $\mathbf{R}^k$. This happens when all masses in Eq. (3.22) are the same so that using Eqs. (3.21) and (3.22) we obtain equation for the simplex. In the case of S^2 we can think of z coordinate as deformation retract for S^2. One can say that S^2 is the *inflated symplectic manifold* corresponding to the segment $[-1, 1]$, i.e. $S^2 \sim R \times S^1$. Accordingly, we can say that $\mathbf{CP}^k \sim \Delta_k \times S^1 \times \cdots S^1$. The Darboux coordinates $\{t_1, \ldots, t_k; \phi_1, \ldots, \phi_k\} \to \{t_1^{1/2} e^{i\phi_1}, \ldots, t_k^{1/2} e^{i\phi_k}, \sqrt{1 - \sum_i t_i}\} \equiv \{z_1, \ldots, z_k, z_{k+1}\}$, provided that $t_1 + \cdots + t_{k+1} = 1$. These results are in accord with Eq. (3.21). Such a choice of coordinates realizes $\mathbf{CP}^k$ as the space of equivalence classes

$$|z_1|^2 + \cdots + |z_k|^2 + |z_{k+1}|^2 = 1, \quad z_i \sim e^{i\phi_i} z_i,\ i = 1 - k. \tag{3.24}$$

of the points lying on the sphere S^{2k+1} in $\mathbf{C}^{k+1}$ (we used this realization of $\mathbf{CP}^k$ already in Part II). In accord with previous subsection, it is the reduced phase space (the reduced symplectic manifold M_{red}) for our dynamical system.

Following Section 1, it is of interest to consider the inflated simplex $n\Delta_k$ living on the lattice $\mathbf{Z}^k$. Accordingly, we can consider the associated with it the inflated symplectic manifold $\mathcal{M}$. The indices of critical points of such a manifold produce its Euler characteristic χ. Irrespective to locations of critical points on such a manifold, the point $\mathbf{t}$ should have coordinates such that $t_1 + \cdots + t_k = n$. If $\mathcal{P}$ is the rational polytope these coordinates should be some integers. Accordingly, one has to count the number of solutions to the equation $t_1 + \cdots + t_k = n$ in order to determine the number $p(k, n)$ of such critical points. This number we know already since it is given by Eq. (1.2). Accordingly, for physically interesting case associated with our interpretation of the Veneziano amplitudes we obtain, $p(k, n) = \chi$. These rather simple arguments are useful to compare with extremely sophisticated proofs of the same result using methods of algebraic geometry, e.g. see Refs. [40–42]. These methods are of importance however in case if one is interested in computation of some observables as it is done earlier, for example, for the Witten–Kontsevich model [51]. More on this will be said below and in Part IV. Obtained results provide us with tools needed for symplectic treatment of the Veneziano amplitudes and for restoration of generating function associated with these amplitudes. This is accomplished in the next section.

[20] Incidentally, if following Delzant [67], we divide the number of polyhedral vertices by factor of 2, then using Eq. (1.7) with $2m$ replaced by 1 we shall reobtain the result $\chi = 2$. More formally, we can say that the cardinality $|G| = \frac{1}{2} \dim \mathcal{M}$. That this is indeed the case in general was proven by Delzant.

4. Exact solution of the Veneziano model: symplectic treatment

4.1. The moment map, the Duistermaat–Heckman formula and the Khovanskii–Pukhlikov correspondence

4.1.1. General remarks

We have mentioned already number of times mathematical connections between the Veneziano amplitudes (and the Veneziano partition function associated with them) and dynamical systems. We would like to summarize these results now. First, already in Part I we emphasized that the development in this series of work is motivated in part by two major observations. These are: (a) the unsymmetrized Veneziano amplitude can be looked upon as the Laplace transform of the partition function obtained by quantization of *finite* set of harmonic oscillators as described in the work by Vergne [3], (b) the unsymmetrized Veneziano amplitude can be interpreted as one of the periods associated with homology cycles on the variety of Fermat-type. These observations are sufficient for development of both symplectic and supersymmetric approaches leading to restoration of the underlying physical model producing the Veneziano-like amplitudes. In Part II we strengthened these observations by invoking theorems by Solomon and Ginzburg (Theorems 2.2 and 2.5, respectively). Also, in Part II using results by Shepard and Todd and Serre we provided enough evidence for the Veneziano partition function to be supersymmetric. Using these results we obtained exact solution for the Veneziano model, i.e. we have obtained the partition/generating function for this model whose observables are unsymmetrized Veneziano amplitudes. In this work we provided additional details directing us towards alternative (symplectic) interpretation of this partition function. These include: (a) zeta function by Ruelle, (b) method of coadjoint orbits and (c) the moment map. Connections between supersymmetric and symplectic descriptions can be deduced using well written monograph by Berline, Getzler and Vergne [70]. In view of this, to avoid excessive size of our paper, it is sufficient to emphasize only things of immediate relevance. In particular, we would like to discuss now the Duistermaat–Heckman formula.

4.1.2. The Duistermaat–Heckman formula

Although the description of the Duistermaat–Heckman (D–H) formula can be found in many places, we would like to discuss it now in connection with earlier obtained results. To this purpose, using Subsection 3.4. let us consider once again the simplest dynamical model discussed there. The volume form Ω for this model is given by $\Omega := d\theta \wedge \mathrm{d}z$ so that $\int_{S^2} \Omega = 4\pi$ as expected. With help of this form the equation for the moment map can be obtained. According to the standard rules [39,61], given that $\xi = \frac{\partial}{\partial\theta}$, we obtain

$$i(\xi)\Omega = \mathrm{d}z. \tag{4.1}$$

The Hamiltonian $\mathcal{H}$, i.e. the moment map, is given in this case by $\mathcal{H} = z$. Consider now the integral $I(\beta)$ of the type

$$I(\beta) = \int_{\mathcal{M}_{\mathrm{red}}} \tilde{\Omega} \exp(-\beta\mathcal{H}) = \frac{1}{\beta}(\exp(\beta) - \exp(-\beta)). \tag{4.2}$$

In this integral the reduced phase space is $\mathcal{M}_{\mathrm{red}} = S^2/S^1$ so that $\tilde{\Omega} = \mathrm{d}z$ and, as before, $z \in [-1, 1]$. Eq. (4.2) is essentially the D–H formula! We would like to explain this fact in some detail. In view of the results of Section 3.3 we know that the moment map $\mathcal{H}$ achieves its extrema at the vertices of $\mathcal{P}$. Since in our case $\mathcal{P}$ is the segment $[-1, 1]$, indeed, $\mathcal{H}$ achieves its extrema at both 1 and -1 so that the right-hand side of Eq. (4.2) is in fact the sum over the vertices of $\mathcal{P}$

taken with the appropriate weights. The D–H formula provides exactly the same answer. Indeed let $M \equiv M^{2n}$ be a compact symplectic manifold equipped with the momentum map $\Phi : M \to \mathbf{R}$ and the (Liouville) volume form $\mathrm{d}V = (\frac{1}{2\pi})^n \frac{1}{n!}\Omega^n$. According to the Darboux theorem, the two-form Ω can be presented locally as: $\Omega = \sum_{l=1}^{n} \mathrm{d}q_l \wedge \mathrm{d}p_l$. Suppose that such a manifold has isolated fixed points p belonging to the fixed point set $\mathcal{V}$ associated with the isotropy subgroup G (Definition 3.5) acting on M. Then, in its most general form, the D–H formula can be written as [39,61]

$$\int_M \mathrm{d}V e^{\Phi} = \sum_{p\in\mathcal{V}} \frac{e^{\Phi(p)}}{\prod_j a_{j,p}} \tag{4.3}$$

where $a_{1,p}, \ldots, a_{n,p}$ are the weights of the linearized action of G on T_pM. Using the Morse theory, Atiyah [62] and others [61] have demonstrated that it is sufficient to keep terms up to quadratic in the expansion of Φ around given p. In such a case the moment map looks exactly like that given in Eq. (3.22). Moreover, the coefficients $a_{1,p}, \ldots, a_{n,p}$ are just "masses" m_i in Eq. (3.22). We put quotation marks around masses since they can be both positive and negative. With these remarks, it should be obvious that Eq. (4.2) is the D–H formula. It should be noted that although in Eq. (4.3) the space M is not reduced, Eq. (4.2) can be written without requirement of reduction as well. For this it is sufficient to consider in Eq. (4.2) the form $\Omega = \frac{1}{2\pi} d\theta \wedge \mathrm{d}z$. Hence, indeed, Eq. (4.2) *is* the D–H formula. Consider now the limiting case $\beta \to 0^+$ of Eq. (4.2). Then, we obtain

$$I(\beta \to 0^+) = 2. \tag{4.4}$$

But 2 is the *Euclidean volume* of the polytope $\mathcal{P}$, in our case, the length of the segment $[-1, 1]$. This is in accord with general result obtained by Atiyah [60]. Now we would like to generalize this apparently trivial result in several directions. First, we would like to blow up the sphere so that its diameter would be $2m$. Second, we would like to consider a collection of such spheres with respective diameters $2m_i$, $i = 1 - d$. For such a collection we can consider two situations: (a) the total energy $\mathcal{E}$ for the Hamiltonian $\mathcal{H} = \sum_i z_i$ is not conserved and (b) the total energy is conserved, e.g. see Eq. (3.22). The first case is nonphysical but, apparently, is relevant for theory of singularities of differentiable maps and is related to the computation of the Milnor number. This issue was discussed in our earlier work, Ref. [71]. The second case is physically relevant. Hence, we would like to discuss it in some detail. Both cases can be found as exercises on page 50 in the book by Guillemin [38]. In discussing the second case both Guillemin [38], and Audin [61], notice that the action for the torus $T^d = S^1 \times \cdots \times S^1$ on such Hamiltonian system is diagonal (e.g. Section 3) and is made of d-tuples $(e^{i\theta_1}, \ldots, e^{i\theta_d})$ subject to the constraint $e^{i\theta_1}, \ldots, e^{i\theta_d} = 1$. This constraint is actually the Veneziano condition discussed in Section 3, e.g. see Eq. (3.14).

Based on the information just mentioned, we would like to be more specific now. To this purpose, following Vergne [72] and Brion [6] we would like to consider the simplest nontrivial case of the integral of the form

$$I(k) = \int_{k\Delta} \mathrm{d}x_1 \mathrm{d}x_2 \exp\{-(y_1x_1 + y_2x_2)\}, \tag{4.5}$$

where $k\Delta$ is dilated standard simplex with coefficient of dilation k. Following these authors, calculation of this integral can be done exactly with the result

$$I(k) = \frac{1}{y_1y_2} + \frac{e^{-ky_1}}{y_1(y_1 - y_2)} + \frac{e^{-ky_2}}{y_2(y_2 - y_1)}. \tag{4.6}$$

As in earlier case of Eq. (4.2), the obtained result fits the D–H formula, Eq. (4.3), and, as before, in the limit: $y_1, y_2 \to 0$, some calculation produces the anticipated result: $Volk\Delta = k^2/2!$, in accord with Eq. (4.4). In view of results of Parts I, II and this work, this integral is of relevance to calculation of Veneziano amplitudes (and it does have symplectic meaning !): the standard simplex Δ in the present case is just the deformation retract for the Fermat (hyper)surface associated with calculation of the Veneziano (or Veneziano-like) amplitudes. The relevance of this integral to the Veneziano amplitude is far from superficial as we would like to discuss now.

4.1.3. The Khovanskii–Pukhlikov correspondence and calculation of χ

The Khovanskii–Pukhlikov correspondence can be understood based on the following generic example. Following Ref. [73] we would like to compare the integral

$$I(z) = \int_s^t \mathrm{d}x e^{zx} = \frac{e^{tz}}{z} - \frac{e^{sz}}{z} \text{ with the sum } S(z) = \sum_{k=s}^{t} e^{kz} = \frac{e^{tz}}{1-e^{-z}} + \frac{e^{sz}}{1-e^{z}}, \tag{4.7}$$

where Eq. (1.4) was used for calculation of $S(z)$.

One can pose a problem: is there way to transform the integral I into the sum S? Clearly, we are interested in such a transform in view of the fact that the Ehrhart polynomial computes the number of lattice points of the dilated polytope while the D–H integral can be used only for calculation of the *Euclidean* volumes of such polytopes as we just demonstrated on simple examples. The positive answer to the above question was found by Khovanskii and Pukhlikov [74] and refined by many others, e.g. see Ref. [73]. Before discussing their work, we would like to write down the discrete analog of the result, Eq. (4.6). It is given by

$$\begin{aligned} S(k) &= \frac{1}{1-e^{-y_1}}\frac{1}{1-e^{-y_2}} + \frac{1}{1-e^{y_1}}\frac{e^{-ky_1}}{1-e^{y_1-y_2}} + \frac{1}{1-e^{y_2}}\frac{e^{-ky_2}}{1-e^{y_2-y_1}} \\ &= \sum_{(l_1,l_2)\in k\Delta} \exp\{-(y_1 l_1 + y_2 l_2)\}. \end{aligned} \tag{4.8}$$

This result can be obtained rather straightforwardly using Brion's formula for the generating function for polytopes. It is given by earlier discussed Eq. (1.11) and, hence, it is in complete accord with this more general equation. Some computational details can be found in the monograph by Barvinok [7]. Following his exposition, we would like to discuss some physics behind these formal calculations. For this we need to use the definition of the monoid S_σ, Eq. (3.3), introduced earlier. In view of the Remark 9.9 (Part II) the set $a_1, \ldots, a_k$ forms a basis of the vector space V so that the monoid S_σ defines a *rational* polyhedral cone σ. Thanks to the theorem by Brion [6,7] the generating function in the left-hand side of Eq. (1.11) can be conveniently rewritten as

$$f(\mathcal{P}, \mathbf{x}) = \sum_{\mathbf{m}\in\mathcal{P}\cap\mathbf{Z}^d} \mathbf{x}^{\mathbf{m}} = \sum_{\sigma\in Vert\,\mathcal{P}} \mathbf{x}^{\boldsymbol{\sigma}} \tag{4.9a}$$

so that for the *dilated* polytope it reads as follows

$$f(k\mathcal{P}, \mathbf{x}) = \sum_{\mathbf{m}\in k\mathcal{P}\cap\mathbf{Z}^d} \mathbf{x}^{\mathbf{m}} = \sum_{i=1}^{n} \mathbf{x}^{\mathbf{k}\mathbf{v}_i} \sum_{\sigma_i} \mathbf{x}^{\boldsymbol{\sigma}_i}. \tag{4.9b}$$

In the last formula the summation is taking place over all vertices whose location is given by the vectors from the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. This means that in actual calculations one can first calculate the contributions coming from the cones σ_i of the undilated (original) polytope $\mathcal{P}$ and only then one can use this equation in order to get the result for the dilated polytope. Let us apply these general rules to our specific problem of computation of $S(k)$ in Eq. (4.8). We have our simplex with vertices in $x-y$ plane given by the vector set $\{\mathbf{v}_1=(0,0), \mathbf{v}_2=(1,0), \mathbf{v}_3=(0,1)\}$ where we have written the x coordinate first. For this case we have three cones: $\sigma_1=l_2\mathbf{v}_2+l_3\mathbf{v}_3$; $\sigma_2=\mathbf{v}_2+l_1(-\mathbf{v}_2)+l_2(\mathbf{v}_3-\mathbf{v}_2)$; $\sigma_3=\mathbf{v}_3+l_3(\mathbf{v}_2-\mathbf{v}_3)+l_1(-\mathbf{v}_3)$; $\{l_1,l_2,l_3\}\in\mathbf{Z}_+$. In writing these expressions for the cones we have taken into account that, according to Brion, when making calculations the apex of each cone should be chosen as the origin of the coordinate system. Calculation of contributions to generating function coming from σ_1 is the most straightforward. Indeed, in this case we have $\mathbf{x}=x_1x_2=e^{-y_1}e^{-y_2}$. Now, the symbol $\mathbf{x}^{\boldsymbol{\sigma}}$ in Eqs. (4.9) should be understood as follows. Since σ_i , $i=1-3$, is actually a vector, it has components, like those for $\mathbf{v}_1$, etc. We shall write therefore $\mathbf{x}^{\boldsymbol{\sigma}}=x_1^{\sigma(1)},\ldots,x_d^{\sigma(d)}$, where $\sigma(i)$ is the i-th component of such a vector. Under these conditions calculation of the contributions from the first cone with the apex located at (0,0) is completely straightforward

$$\sum_{(l_2,l_3)\in Z_+^2} x_1^{l_2}x_2^{l_3}=\frac{1}{1-e^{-y_1}}\frac{1}{1-e^{-y_2}} \tag{4.10}$$

since it is reduced to the computation of the infinite geometric progressions. But physically, the above result can be looked upon as a product of two partition functions for two harmonic oscillators whose ground state energy was discarded. By doing the rest of calculations in the way just described we reobtain $S(k)$ from Eq. (4.8) as required. This time, however, we know that the obtained result is associated with the assembly of harmonic oscillators of frequencies $\pm y_1$, $\pm y_2$ and $\pm(y_1-y_2)$ whose ground state energy is properly adjusted. The "frequencies" (or masses) of these oscillators are coming from the Morse-theoretic considerations for the moment maps associated with the critical points of symplectic manifolds as explained in the paper by Atiyah [62]. These masses enter into the "*classical*" D–H formula. It is just a classical partition function for a system of such described harmonic oscillators living in phase space containing singularities. The D–H classical partition function, Eq. (4.6), has its quantum analog, Eq. (4.8), just described. The ground state for such a quantum system is degenerate with degeneracy being described by the Kostant multiplicity formula. To calculate this degeneracy would require us to study the limiting case: $y_1, y_2\to 0$ of Eq. (4.8) for $S(k)$. Surprisingly, unlike the continuum case studied in the previous subsection, calculation of number of points belonging to the dilated simplex $k\Delta$ (which is equivalent to the calculation of the Kostant multiplicity formula or, which is the same, to the computation of the Ehrhart polynomial or to the Euler characteristic χ of the associated projective toric variety) is very nontrivial in the present case. It is facilitated by the observation that in the limit $s\to 0$ the following expansion can be used

$$\frac{1}{1-e^{-s}}=\frac{1}{s}+\frac{1}{2}+\frac{s}{12}+O(s^2). \tag{4.11}$$

Rather lengthy calculation involving this expansion produces in the end the anticipated result for the Ehrhart polynomial:

$$|k\Delta\cap Z^2|=P(k,2)=\frac{k^2}{2}+\frac{3}{2}k+1. \tag{4.12}$$

Obtained results and their interpretations are in formal accord with those by Vergne [3]. In her work no details (like those presented above) or physical applications are given however. At the same time, the results obtained thus far apparently are not in agreement with earlier obtained major result, Eq. (1.1). Fortunately, the situation can be corrected with help of the following theorem by Barvinok [75].

Theorem 4.1. *For the fixed lattice of dimensionality d there exist a polynomial time algorithms which, for any given rational polytope* $\mathcal{P}$*, calculate the generating function* $f(\mathcal{P}, x)$ *with the result:*

$$f(\mathcal{P}, x) = \sum_{\mathbf{m}\in\mathcal{P}\cap\mathbf{Z}^d} \mathbf{x}^{\mathbf{m}} = \sum_{i\in I} \epsilon_i \frac{\mathbf{x}^{p_i}}{(1-\mathbf{x}^{a_{i\,1}})\ldots(1-\mathbf{x}^{a_{i\,d}})}, \tag{4.13}$$

where $\epsilon \in \{-1, 1\}$, $p_i, a_{i\,j} \in \mathbf{Z}^d$ *and* $a_{i\,j} \neq 0\,\forall i, j$*. In fact,* $\forall i$ *the set* $a_{i\,1}, \ldots, a_{i\,d}$ *forms a basis of* $\mathbf{Z}^d$ *and I is the set* $\{1, \ldots, n\}$ *labeling the vertices of* $\mathcal{P}$*.*

Remark 4.2. It is easy to check this result using Eq. (1.1) for n (or d) equal to 2 and comparing it with $S(k)$ from Eq. (4.8). It should be noted however, that Eq. (1.1) was obtained using some kind of combinatorial and supersymmetric arguments as explained in Part II while Eq. (4.8) is obtained exclusively based on use of the bosonic formalism. It should be clear that both approaches leading to the design of new model reproducing the Veneziano and Veneziano-like amplitudes can be used in principle since they are essentially equivalent in view of the earlier mentioned Ref. [70].

At this point, finally, we are ready do discuss the Khovanskii–Pukhlikov correspondence. It should be considered as alternative to the method of coadjoint orbits discussed in Section 3.2. Naturally, both methods are in agreement with each other with respect to final results. Following Refs. [38,72,73] we introduce the Todd operator (transform) via

$$Td(z) = \frac{z}{1-e^{-z}}. \tag{4.14}$$

In view of Eq. (4.7), it can be demonstrated [73] that

$$Td\left(\frac{\partial}{\partial h_1}\right) Td\left(\frac{\partial}{\partial h_2}\right)\left(\int_{s-h_1}^{t+h_2} e^{zx}\,\mathrm{d}x\right)|_{h_1=h_1=0} = \sum_{k=s}^{t} e^{kz}. \tag{4.15}$$

This result can be now broadly generalized following ideas of Khovanskii and Pukhlikov [66]. In particular, the relation

$$Td\left(\frac{\partial}{\partial \mathbf{z}}\right)\exp\left(\sum_{i=1}^{n} p_i z_i\right) = Td(p_1, \ldots, p_n)\exp\left(\sum_{i=1}^{n} p_i z_i\right) \tag{4.16}$$

happens to be the most useful. Applying it to

$$i(x_1, \ldots, x_k; \xi_1, \ldots, \xi_k) = \frac{1}{\xi_1, \ldots, \xi_k}\exp\left(\sum_{i=1}^{k} x_i\xi_i\right) \tag{4.17}$$

we obtain

$$s(x_1, \ldots, x_k; \xi_1, \ldots, \xi_k) = \frac{1}{\prod_{i=1}^{k}(1-\exp(-\xi_i))}\exp\left(\sum_{i=1}^{k} x_i\xi_i\right). \tag{4.18}$$

This result should be compared now with the individual terms on the right-hand side of Eq. (1.11) on one hand and with the individual terms on the right-hand side of Eq. (4.3) on another. Evidently, with help of the Todd transform the exact “classical” results for the D–H integral are transformed into the “quantum” Weyl character formula.

We would like to illustrate these general observations by comparing the D–H result, Eq. (4.6), with the Weyl character formula (e.g. see Eq. (1.11)), Eq. (4.8). To this purpose we need to use already known data for the cones $\sigma_i, i = 1 - 3$, and the convention for the symbol $\mathbf{x}^\sigma$. In particular, for the first cone we have already: $\mathbf{x}^{\sigma_1} = x_1^{l_1} x_2^{l_2} = [\exp(l_1 y_1)] \cdot [\exp(l_2 y_2)]$.[21] Now we assemble the contribution from the first vertex using Eq. (4.6). We obtain, $[\exp(l_1 y_1)] \cdot [\exp(l_2 y_2)]/y_1 y_2$. Using the Todd transform we obtain as well

$$Td\left(\frac{\partial}{\partial l_1}\right) Td\left(\frac{\partial}{\partial l_2}\right) \frac{1}{y_1 y_2}[\exp(l_1 y_1)] \cdot [\exp(l_2 y_2)] \mid_{l_1=l_2=0} = \frac{1}{1-e^{-y_1}} \frac{1}{1-e^{-y_2}}. \tag{4.19}$$

Analogously, for the second cone we obtain: $\mathbf{x}_2^\sigma = e^{-ky_1} e^{-l_1 y_1} e^{-l_2(y_1-y_2)}$ so that use of the Todd transform produces

$$Td\left(\frac{\partial}{\partial l_1}\right) Td\left(\frac{\partial}{\partial l_2}\right) \frac{1}{y_1 (y_1 - y_2)} e^{-ky_1} e^{-l_1 y_1} e^{-l_2(y_1-y_2)} \mid_{l_1=l_2=0} = \frac{1}{1-e^{y_1}} \frac{e^{-ky_1}}{1-e^{y_1-y_2}}, \tag{4.20}$$

etc.

In Section 3.4, we sketched ideas behind calculations of Euler characteristic χ. It is instructive in the light of just obtained results to reobtain χ.

To accomplish the task is actually not difficult since it is based on the information we have presented already. Indeed, by looking at the last two equations it makes sense to rewrite formally the partition function, Eq. (4.5), in the following symbolic form

$$I(k, \mathbf{f}) = \int_{k\Delta} \mathrm{d}\mathbf{x} \exp(-\mathbf{f} \cdot \mathbf{x}) \tag{4.21}$$

valid for any finite dimension d. Since we have performed all calculations explicitly for two-dimensional case, for the sake of space, we only provide the idea behind such type of calculation for any d.[22] In particular, using Eq. (4.3) we can rewrite this integral formally as follows:

$$\int_{k\Delta} \mathrm{d}\mathbf{x} \exp(-\mathbf{f} \cdot \mathbf{x}) = \sum_p \frac{\exp(-\mathbf{f} \cdot \mathbf{x}(p))}{\prod_i^d h_i^p(\mathbf{f})}. \tag{4.22}$$

Applying the Todd operator (transform) to both sides of this formal expression and taking into account Eqs. (4.19), (4.20) (providing assurance that such an operation indeed is legitimate and makes sense) we obtain

$$\begin{aligned} &\int_{k\Delta} \mathrm{d}\mathbf{x} \prod_{i=1}^d \frac{x_i}{1-\exp(-x_i)} \exp(-\mathbf{f} \cdot \mathbf{x}) \\ &= \sum_{\mathbf{v} \in Vert\, \mathcal{P}} \exp\{\langle \mathbf{f} \cdot \mathbf{v} \rangle\} \left[\prod_{i=1}^d (1 - \exp\{-h_i^v(\mathbf{f}) u_i^v\})\right]^{-1} = \sum_{\mathbf{x} \in \mathcal{P} \cap \mathbf{Z}^d} \exp\{\langle \mathbf{f} \cdot \mathbf{x} \rangle\} \end{aligned} \tag{4.23}$$

[21] To obtain correct results we needed to change signs in front of l_1 and l_2. The same should be done for other cones as well.

[22] Mathematically inclined reader is encouraged to read paper by Brion and Vergne [76], where all missing details are scrupulously presented.

where the last equation was written in view of Eq. (1.11). From here, it is clear that in the limit: $\mathbf{f} = 0$ we reobtain back χ.

4.2. From Riemann–Roch–Hirzebruch to Witten and Lefschetz via Atiyah and Bott

As it was noticed already by Khovanskii and Puklikov [74] and elaborated by others, e.g. see Ref. [76], in the limit: $\mathbf{f} = 0$ the integral in the left-hand side of Eq. (4.23) can be associated with the Hirzebruch–Grotendieck–Riemann–Roch formula for the Euler characteristic $\chi(E)$. In standard notations [76,77] it is given by

$$\chi(E) = \int_X ch(E) \cdot Td(TX), \tag{4.24}$$

where E is a vector bundle over the variety X, $ch(E)$ the Chern character of E, and $Td(TX)$ is the Todd class of the tangent bundle TX of X. This formula is too formal to be used immediately. The mathematical formalism of equivariant cohomology is needed for actual calculations connecting Eq. (4.24) with the left-hand side of Eq. (4.23). It was developed in the classical paper by Atiyah and Bott [25] inspired by earlier work by Witten [26] on supersymmetry and Morse theory. In this work we shall use only a small portion of their results. A very pedagogical exposition of the results by Atiyah and Bott can be found in the monograph by Guillemin and Sternberg [78] also containing helpful additional supersymmetric information.

We begin our discussion with the following observations. Earlier, in Section 3.2, we introduced the Kirillov–Kostant symplectic two-form ω_x. We noticed that this form is defined everywhere outside the set of critical points of symplectic manifold $\mathcal{M}$. The simplest example of the symplectic two-form was given in Section 3.4 for the case of two-sphere S^2 where it coincides with the volume form $\Omega = d\phi \wedge \mathrm{d}z$ for which $\int_{S^2} \Omega = 4\pi$. At the same time, the *symplectic* volume form is given by $\Omega/2\pi$ so that the integral over S^2 becomes equal to 2. This fact reminds us about the Gauss-Bonnet theorem for the two-sphere which is prompting us to associate the two-form $\Omega/2\pi$ with the curvature two-form. To make things more interesting we recall some facts from the differential analysis on complex manifolds as described, for example, in the book by Wells [55]. From this reference we find that the first Chern class $c_1(E)$ of the E vector bundle over the sphere S^2 is given by

$$c_1(E) = \frac{i}{\pi} \frac{\mathrm{d}z \wedge d\bar{z}}{(1+|z|^2)^2} = \frac{2}{\pi} \frac{\rho \mathrm{d}\rho \mathrm{d}\phi}{(1+\rho^2)^2} \tag{4.25}$$

so that $\int_{S^2} c_1(E) = 2$. Next, let us recall that *any* Kähler manifold is symplectic [61] and that for any Kähler manifold the second fundamental form Ω can be written locally as $\Omega = \frac{i}{2} \sum_{ij} h_{ij}(z) \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$ so that $h_{ij}(z) = \delta_{ij} + O(|z|^2)$. Hence, any symplectic volume form can be rewritten in terms of just described form Ω. The form Ω is closed but not exact. Evidently, up to a constant, $c_1(E)$ in Eq. (5.2) coincides with the standard Kähler two-form. In view of the Gauss–Bonnet theorem, it is not exact. An easy calculation shows that the form $d\phi \wedge dz$ can also be brought to the standard Kähler two-form (again up to a constant). Moreover, for the Hamiltonian of planar harmonic oscillator discussed in Section 3.3. the standard symplectic two-form Ω can be written in several equivalent ways

$$\Omega = \mathrm{d}x \wedge \mathrm{d}y = r\,\mathrm{d}r \wedge \mathrm{d}\theta = \frac{1}{2}\mathrm{d}r^2 \wedge \mathrm{d}\theta = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{4.26}$$

and is certainly Kählerian. For collection of k harmonic oscillators the symplectic two-form Ω is given, as usual, by $\Omega = \sum_{i=1}^{k} \mathrm{d}x_i \wedge \mathrm{d}y_i = \frac{i}{2} \sum_{i=1}^{k} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$ so that its n-th power is given by

$\Omega^n = \Omega \wedge \Omega \wedge \cdots \wedge \Omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \mathrm{d}x_n \wedge \mathrm{d}y_n$. In view of these results, it is convenient to introduce the differential form

$$\exp \Omega = 1 + \Omega + \frac{1}{2!}\Omega \wedge \Omega + \frac{1}{3!}\Omega \wedge \Omega \wedge \Omega + \cdots. \tag{4.27}$$

By design, this expansion will have only k terms. Our earlier discussion of the moment map in Section 3.3. suggests that just described case of the collection of harmonic oscillators is generic since its existence is guaranteed by the Morse theory as discussed by Atiyah [62].[23] In view of Eq. (4.25) such an expansion can be formally associated with the total Chern class. Hence, we shall associate $\exp \Omega$ with the total Chern class. Since all symplectic manifolds we considered earlier possess singularities the standard homology and cohomology theories should be replaced by equivariant ones as explained by Atiyah and Bott [25]. To this purpose we observe that in the absence of singularities the symplectic two-form Ω is always closed, i.e. $d\Omega = 0$. In case of singularities, one should replace the exterior derivative d by $\tilde{d} = d + i(\xi)$[24] while changing Ω to $\Omega - \mathbf{f} \cdot \mathbf{x}$ in notations of Eq. (4.21). The D–H integral, Eq. (2.22) can be formally rewritten now as

$$\int_{k\Delta} \mathrm{d}\mathbf{x} \exp(-\mathbf{f} \cdot \mathbf{x}) = \int_{k\Delta} \exp(\tilde{\Omega}) \tag{4.28}$$

where $\tilde{\Omega} = \Omega - \mathbf{f} \cdot \mathbf{x}$. The form $\tilde{\Omega}$ is equivariantly closed. Indeed, since $\tilde{d}\tilde{\Omega} = d\Omega + i(\xi)\Omega - \mathbf{f} \cdot \mathbf{dx}$ then, in view of Eq. (4.1), $i(\xi)\Omega - \mathbf{f} \cdot \mathbf{dx} = 0$ by design, while $d\Omega = 0$ everywhere, except at singularities (critical points) where $\Omega = 0$ (as discussed in Section 3.2). Hence, $\tilde{d}\tilde{\Omega} = 0$ as required. Since Ω can be identified with the Chern class one should identify $\mathbf{f} \cdot \mathbf{x}$ with the Chern class as well, i.e. $\mathbf{f} \cdot \mathbf{x} \equiv \sum_{i=1}^{d} f_i c_i$ where we took into account that $E = L_1 \oplus \cdots L_d$ because $\mathbf{C}^d = \mathbf{C} \oplus \mathbf{C} \oplus \cdots \oplus \mathbf{C}$ so that L_i is the line bundle associated with $\mathbf{C}_i$. After such an identification Eq. (4.24) can be rewritten as

$$\chi(E) = \int_X e^{\Omega} \cdot \prod_{i=1}^{d} \frac{c_i}{1 - \exp(-c_i)}. \tag{4.29}$$

Obtained result is in agreement with that given in the book by Guillemin [38, p. 60].

In view of the results of Part I, and Theorems 2.2 (by Solomon) and 2.5. (by Ginzburg) of Part II one can achieve more by discussing the intersection cohomology ring of the reduced spaces associated with the D–H measures. Since in Part I we noticed already that the Veneziano amplitudes can be formally associated with the period integrals for the Fermat (hyper)surfaces $\mathcal{F}$ and since such integrals can be interpreted as intersection numbers between the cycles on $\mathcal{F}$, one can formally rewrite the precursor to the Veneziano amplitude (as discussed in Part I) as

$$I = \left(\frac{-\partial}{\partial f_1}\right)^{r_1} \cdots \left(\frac{-\partial}{\partial f_d}\right)^{r_d} \int_{\Delta} \exp(\tilde{\Omega}) \mid_{f_i = 0 \forall i} = \int_{\Delta} \mathrm{d}\mathbf{x} (c_1)^{r_1} \cdots (c_d)^{r_d} \tag{4.30}$$

provided that $r_1 + \cdots + r_d = n$. In such a language, the problem of calculation of the Veneziano amplitudes using generating function, Eq. (4.28), becomes mathematically almost equivalent to earlier considered calculations related to the Witten–Kontsevich model discussed earlier in Ref.

[23] In view of earlier discussed examples, we are interested only in the rotationally invariant observables, this means that the θ (or ϕ) dependence in the two-form, Eq. (5.3), can be dropped which is equivalent to considering only the reduced phase space. This is meaningful both mathematically and physically. Details can be found in Ref. [33, pp. 65–71].

[24] E.g. see Eq. (4.1).

[42]. This circumstance will be exploited further in Part IV. Obtained results provide complete symplectic solution of the Veneziano model.

As it was noticed by Atiyah and Bott [25], the replacement of exterior derivative d by $\tilde{d} = d + i(\xi)$ was inspired by earlier work by Witten on supersymmetric formulation of quantum mechanics and Morse theory, Ref. [26]. Such an observation allows us to discuss calculation of χ and, hence, the Veneziano amplitudes using the supersymmetric formalism developed by Witten. The traditional way of developing Witten's ideas is discussed in detail in earlier mentioned monograph, Ref. [70]. Its essence is well summarized by Guillemin [38]. Following this reference we notice that χ is equal to the dimension $Q = Q^+ - Q^-$ of the quantum Hilbert space associated with the classical system described by the (moment map) Hamiltonian as explained earlier in this section. To describe quantum spaces associated with Q^+ and Q^- we need to remind our readers of several facts from the differential analysis on complex manifolds already discussed in Part II.

We begin with the following observations. Let X be the complex Hermitian manifold and let $\mathcal{E}^{p+q}(X)$ denote the complex-valued differential forms (sections) of type (p, q), $p + q = r$, living on X. The Hodge decomposition insures that $\mathcal{E}^r(X) = \sum_{p+q=r} \mathcal{E}^{p+q}(X)$. The Dolbeault operators ∂ and $\bar{\partial}$ act on $\mathcal{E}^{p+q}(X)$ according to the rule $\partial : \mathcal{E}^{p+q}(X) \to \mathcal{E}^{p+1,q}(X)$ and $\bar{\partial} : \mathcal{E}^{p+q}(X) \to \mathcal{E}^{p,q+1}(X)$, so that the exterior derivative operator is defined as $d = \partial + \bar{\partial}$. Let now $\varphi_p, \psi_p \in \mathcal{E}^p$. By analogy with traditional quantum mechanics we define (using Dirac's notations) the inner product

$$\langle \varphi_p \mid \psi_p \rangle = \int_M \varphi_p \wedge *\bar{\psi}_p, \tag{4.31}$$

where the bar means the complex conjugation and the star $(*)$ means the usual Hodge conjugation. Use of such a product is motivated by the fact that the period integrals, e.g. those for the Veneziano-like amplitudes, and, hence, those given by Eq. (4.30), are expressible through such inner products [55]. Fortunately, such a product possesses properties typical for the finite dimensional quantum mechanical Hilbert spaces. In particular

$$\langle \varphi_p \mid \psi_q \rangle = C\delta_{p,q} \quad \text{and} \quad \langle \varphi_p \mid \varphi_p \rangle > 0, \tag{4.32}$$

where C is some positive constant. With respect to such defined scalar product it is possible to define all conjugate operators, e.g. d^*, etc. and, most importantly, the Laplacians

$$\Delta = dd^* + d^*d, \quad \Box = \partial\partial^* + \partial^*\partial, \quad \bar{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}. \tag{4.33}$$

All this was known to mathematicians *before* Witten's work [26]. The unexpected twist occurred when Witten suggested to extend the notion of the exterior derivative d. Within the de Rham picture (valid for both real and complex manifolds) let M be a compact Riemannian manifold and K be the Killing vector field which is just one of the generators of isometry of M, then Witten suggested to replace the exterior derivative operator d by the extended operator

$$d_s = d + si(K) \tag{4.34}$$

discussed earlier in the context of the equivariant cohomology. Here, s is real nonzero parameter conveniently chosen. Witten argues that one can construct the Laplacian (the Hamiltonian in his formulation) Δ by replacing Δ by $\Delta_s = d_s d_s^* + d_s^* d_s$. This is possible if and only if $d_s^2 = d_s^{*2} = 0$ or, since $d_s^2 = s\mathcal{L}(K)$, where $\mathcal{L}(K)$ is the Lie derivative along the field K, if the Lie derivative acting on the corresponding differential form vanishes. The details are beautifully explained in the much earlier paper by Frankel [68] mentioned earlier in Section 3.3. Atiyah and Bott observed that the multicomponent auxiliary parameter **s** can be identified with earlier introduced

f. This observation provides the link between the symplectic D–H formalism discussed earlier and Witten's supersymmetric quantum mechanics. Looking at Eq. (4.33) and following Refs. [3,38,39,70] we consider the (Dirac) operator $\acute{\partial} = \bar{\partial} + \bar{\partial}^{*}$ and its adjoint with respect to scalar product, Eq. (4.32), then use of the above references suggests that

$$Q = \ker \acute{\partial} - co \ker \acute{\partial}^{*} = Q^{+} - Q^{-} = \chi. \tag{4.35}$$

in accord with Vergne [3]. The results just described provide yet another link between the supersymmetric and symplectic formalisms. Additional details can be found both in Part II and references just cited.

Note added in proof

After this work has been completed and accepted for publication, we became aware of the following two recent papers: arxiv:math.CO/0507163 and arxiv.math.CO/0504231. These papers are not only supporting results presented in the main text, they also provide numerous additional details potentially useful for physical applications. Some of these applications will be discussed in Part IV.

References

[1] A.L. Kholodenko, New string for old Veneziano amplitudes. I. Analytical treatment, J. Geom. Phys. 55 (2005) 50–74.
[2] A.L. Kholodenko, New string for old Veneziano amplitudes. II. Group-theoretic treatment, J. Geom. Phys., doi:10.1016/j.geomphys.2005.10.004.
[3] M. Vergne, Convex polytopes and quanization of symplectic manifolds, Proc. Natl. Acad. Sci. 93 (1996) 14238–14242.
[4] W. Lerche, C. Wafa, N. Warner, Chiral rings in $N = 2$ superconformal theories, Nucl. Phys. B 324 (1989) 427–474.
[5] R. Stanley, Combinatorial reciprocity theorems, Adv. Math. 14 (1974) 194–253.
[6] M. Brion, Points entiers dans les polyedres convexes, Ann. Sci. Ecole Norm. Sup. 21 (1988) 653–663.
[7] A. Barvinok, A Course in Convexity, AMS Publishers, Providence, RI, 2002.
[8] R. Diaz, S. Robins, The Ehrhart polynomial of a lattice polytope, Ann. Math. 145 (1997) 503–518.
[9] R. Stanley, Combinatorics and Commutative Algebra, Birkhäuser, Boston, MA, 1996.
[10] V. Buchchtaber, T. Panov, Torus Actions and Their Applications in Topology and Combinatorics, AMS Publishers, Providence, RI, 2002.
[11] M. Green, J. Schwarz, E. Witten, Superstring Theory, vol. 1, Cambridge University Press, Cambridge, UK, 1987.
[12] V. De Alfaro, S. Fubini, G. Furlan, C. Rossetti, Currents in Hadron Physics, Elsevier Publ. Co., Amsterdam, 1973.
[13] R. Stanley, Invariants if finite groups and their applications to combinatorics, BAMS (New Series) 1 (1979) 475–511.
[14] V. Batyrev, Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori, Duke Math. J. 69 (1993) 349–409.
[15] T. Hibi, Dual polytopes of rational convex polytopes, Combinatorica 12 (1992) 237–240.
[16] N. Ashcroft, D. Mermin, Solid State Physics, Saunders College Press, Philadelphia, 1976.
[17] B. Greene, M. Plesser, Duality in Calabi–Yau moduli space, Nucl. Phys. B 338 (1990) 15–37.
[18] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties, J. Alg. Geom. 3 (1994) 493–535.
[19] G. Veneziano, Construction of crossing-symmetric, Regge behaved, amplitude for linearly rising trajectories, Il Nuovo Chim. 57A (1968) 190–197.
[20] S. Donnachie, G. Dosch, P. Landshoff, O. Nachtmann, Pomeron Physics and QCD, Cambridge University Press, Cambridge, 2002.
[21] P. Collins, An Introduction to Regge Theory and High Energy Physics, Cambridge University Press, Cambridge, 1977.
[22] P. Frampton, Dual Resonance Models, W.A. Benjamin, Inc., Reading, MA, 1974.
[23] S. Mandelstam, Veneziano formula with trajectories spaced by two units, Phys. Rev. Lett. 21 (1968) 1724–1728.

[24] A. Kholodenko, New models for Veneziano amplitudes: combinatorial, symplectic and supersymmetric aspects, Int. J. Geom. Meth. Mod. Phys. 2 (2005) 563–584.
[25] M. Atiyah, R. Bott, The moment map and equivariant cohomology, Topology 23 (1984) 1–28.
[26] E. Witten, Supersymmetry and Morse theory, J. Diff. Geom. 17 (1982) 661–692.
[27] H. Coxeter, Regular Polytopes, The Macmillan Co., New York, 1963.
[28] G. Ziegler, Lectures on Polytopes, Springer-Verlag, Inc., Berlin, 1995.
[29] D. Ruelle, Dynamical Zeta Functions for Piecevise Monotone Maps of the Interval, AMS, Providence, RI, 1994.
[30] M. Atiyah, R. Bott, A Lefschetz fixed point formula for elliptic complexes. I, Ann. Math. 86 (1967) 374–407;
M. Atiyah, R. Bott, A Lefschetz fixed point formula for elliptic complexes. II, Applications. Ann. Math. 88 (1968) 451–491.
[31] I. Gelfand, M. Kapranov, A. Zelevinsky, Discriminants, Resultants and Multidimensional Determinats, Birkhäuser, Inc., Boston, MA, 1994.
[32] N. Bourbaki, Groupes et Algebres de Lie (Chapitre 4–6), Hermann, Paris, 1968.
[33] V. Guillemin, E. Lerman, S. Sternberg, Symplectic Fibrations and Multiplicity Diagrams, Cambridge University Press, Cambridge, UK, 1996.
[34] P. Cartier, On Weil's character formula, BAMS 67 (1961) 228–230.
[35] R. Bott, On induced representations, Proc. Symp. Pure Math. 48 (1988) 1–13.
[36] V. Kac, Infinite Dimensional Lie Algebras, Cambridge University Press, Cambridge, UK, 1990.
[37] R. Feres, Dynamical Systems and Semisimple Groups: An Introduction, Cambridge University Press, Cambridge, UK, 1998.
[38] V. Guillemin, Moment Maps and Combinatorial Invariants of Hamiltonian T^n Spaces, Birkhäuser, Inc., Boston, 1994.
[39] V. Guillemin, V. Ginzburg, Y. Karshon, Moment Maps, Cobordisms and Hamiltonian Group Actions, AMS, Providence, RI, 2002.
[40] W. Fulton, Introduction to Toric Varieties, Ann. Math. Studies 131, Princeton University Press, Princeton, 1993.
[41] G. Ewald, Combinatorial convexity and Algebraic Geometry, Springer-Verlag, Inc., Berlin, 1996.
[42] V. Danilov, The geometry of toric varieties, Russ. Math. Surveys 33 (1978) 97–154.
[43] B. Iversen, The geometry of algebraic groups, Adv. Math. 20 (1976) 57–85.
[44] B. Iversen, H. Nielsen, Chern numbers and diagonalizable groups, J. London Math. Soc. 11 (1975) 223–232.
[45] H. Hopf, Uber die topologie der gruppen-manigfaltigkeiten uber ihre verallgemeinerungen, Ann. Math. 42 (1941) 22–52.
[46] H. Hopf, H. Samelson, Ein satz uber die wirkungesraume geschlossener Liescher gruppen, Commun. Math. Helv. 13 (1941) 240–251.
[47] A. Borel, Linear Algebraic Groups, Springer-Verlag, Inc., Berlin, 1991.
[48] I. Macdonald, Linear Algebraic Groups, LMS Student Texts 32, Cambridge University Press, Cambridge, UK, 1999.
[49] J. Humphreys, Linear Algebraic Groups, Springer-Verlag, Inc., Berlin, 1975.
[50] H. Hiller, Geometry of Coxeter Groups, Pitman Inc., Boston, 1982.
[51] A. Kholodenko, Kontsevich–Witten model from $2+1$ gravity: new exact combinatorial solution, J. Geom. Phys. 43 (2002) 45–91.
[52] F. Knudsen, G. Kempf, D. Mumford, B. Saint-Donat, Toroidal Embeddings I, LNM, 339, Springer-Verlag, Inc., Berlin, 1973.
[53] J.-P. Serre, Algebres de Lie Semi-simples Complexes, Benjamin, Inc., New York, 1966.
[54] J. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer-Verlag, Inc., Berlin, 1972.
[55] R. Wells, Differential Analysis on Complex Manifolds, Springer-Verlag, Inc., Berlin, 1980.
[56] V. Ginzburg, Representation Theory and Complex Geometry, Birkhäuser Verlag, Inc., Boston, 1997.
[57] A. Kirillov, Elements of the Theory of Representations, (in Russian), Nauka, Moscow, 1972.
[58] A. Fomenko, V. Trofimov, Integrable Systems on Lie Algebras and Symmetric Spaces, Gordon and Breach Publishers, New York, 1988.
[59] A. Kholodenko, Use of meanders and train tracks for description of defects and textures in liquid crystals and $2+1$ gravity, J. Geom. Phys. 33 (2000) 23–58.
[60] M. Atiyah, Angular momentum, convex polyhedra and algebraic geometry, Proc. Edinburg Math. Soc. 26 (1983) 121–138.
[61] M. Audin, Torus Actions on Symplectic Manifolds, Birkhäuser, Inc., Boston, 2004.
[62] M. Atiyah, Convexity and commuting Hamiltonians, Bull. London Math. Soc. 14 (1982) 1–15.
[63] A. Schrijver, Combinatorial Optimization. Polyhedra and Efficiency, Springer-Verlag, Inc., Berlin, 2003.
[64] S. Gass, Linear Programming, McGraw-Hill Co., New York 1975.

[65] O. Debarre, Fano Varieties, Higher Dimensional Varieties and Rational Points, Springer-Verlag, Berlin, 2003, pp. 93–132.
[66] V. Guillemin, S. Sternberg, Convexity properties of the moment mapping, Invent. Math. 67 (1982) 491–513.
[67] T. Delzant, Hamiltoniens periodiques et image convexe de l'application moment, Bull. Soc. Math. France 116 (1988) 315–339.
[68] T. Frankel, Fixed points and torsion on Kähler manifolds, Ann. Math. 70 (1959) 1–8.
[69] H. Flaska, Integrable systems and torus actions, in: O. Babelon, P. Cartier, Y. Schwarzbach (Eds.), Lectures on Integrable Systems, World Scientific Pub. Co., Singapore, 1994.
[70] N. Berline, E. Getzler, M. Vergne, Heat Kernels and Dirac Operators, Springer-Verlag, Berlin, 1991.
[71] A. Kholodenko, New Veneziano amplitudes from "old" Fermat (hyper) surfaces, in: C. Benton (Ed.), Trends in Mathematical Physics Research, Nova Science Publ, New York, 2004, pp. 1–94.
[72] M. Vergne, Residue formulae for Verlinde sums, and for number of integral points in convex polytopes, in: E. Mezetti, S. Paycha (Eds.), European Women in Mathematics, World Scientific, Singapore, 2003, pp. 225–284.
[73] M. Brion, M. Vergne, Lattice points in simple polytopes, J. AMS 10 (1997) 371–392.
[74] A. Khovanskii, A. Pukhlikov, A Riemann–Roch theorem for integrals and sums of quasipolynomials over virtual polytopes, St. Petersburg Math. J. 4 (1992) 789–812.
[75] A. Barvinok, K. Woods, Short rational generating functions for lattice point problems, AMS J. 16 (2003) 957–979.
[76] M. Brion, M. Vergne, An Equivariant Riemann–Roch theorem for complete, simplicial toric varieties, J. Reine Angew. Math. 482 (1997) 67–92.
[77] F. Hirzebruch, D. Zagier, The Atiyah–Singer Theorem and Elementary Number Theory, Publish or Perish Inc., Berkeley, CA, 1974.
[78] V. Guillemin, S. Sternberg, Supersymmetry and Equivariant de Rham Theory, Springer-Verlag, Berlin, 1999.